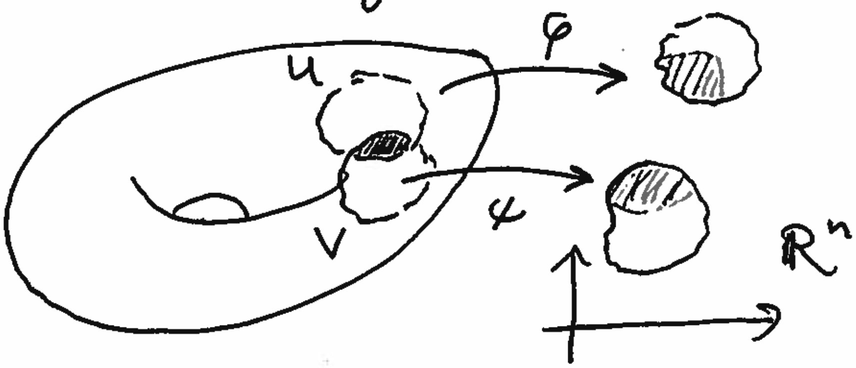
Lecture 3: Smooth maps and diffeomorphisms.

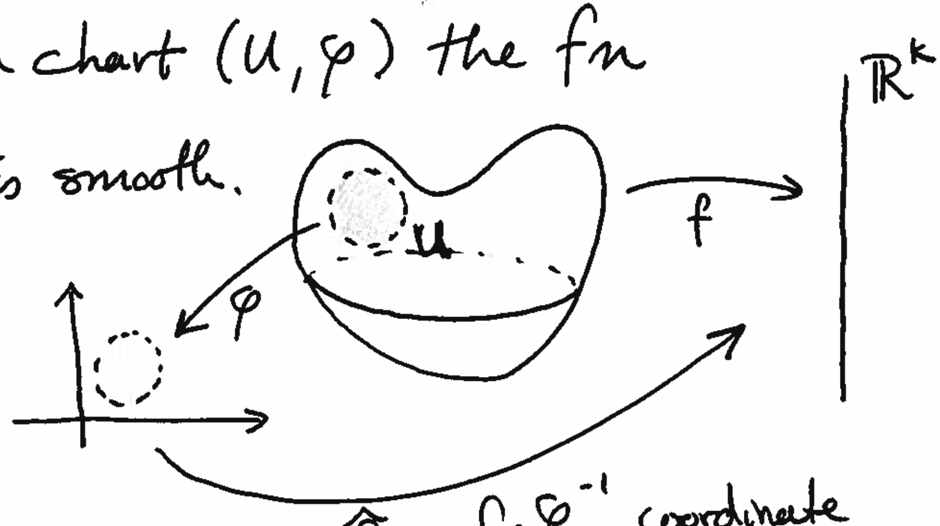
Last time:

Smooth manifold: A topological manifold M with charts covering M so that each pair is compatible, i.e.



$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Smooth fn: M a smooth mfld. A fn $f : M \to \mathbb{R}^k$ is smooth if for every smooth chart (U, φ) the fn $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth.



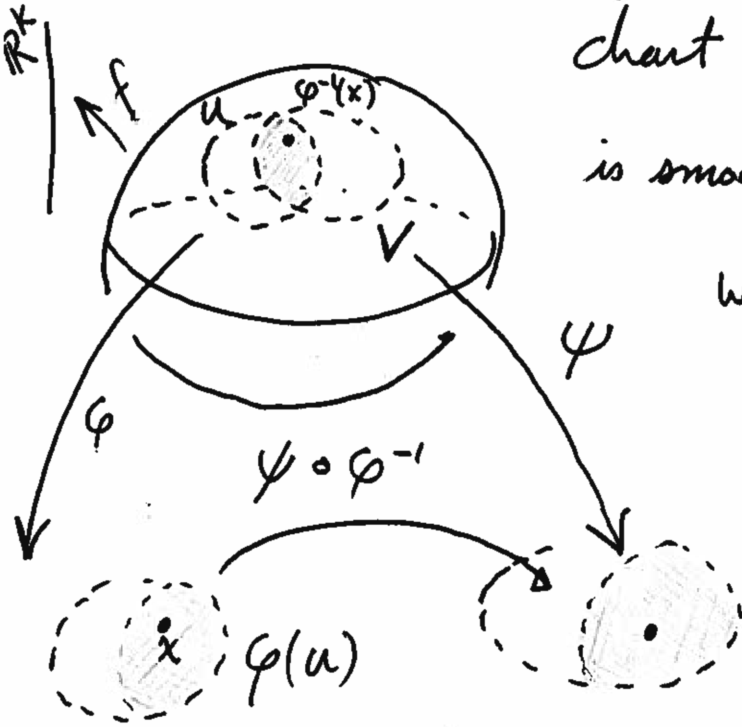
$\hat{f} = f \circ \varphi^{-1}$ coordinate rep'n.

Reminders:

- No class on Monday
- HW #1 due Wednesday, Sept 3.

<u>Lemma</u>: M smooth, $f : M \to \mathbb{R}^k$. If every $p \in M$ is contained in a smooth chart where $\hat{f}$ is smooth, then f is smooth.

<u>Pf</u>: Given an arbitrary smooth chart (U, φ) need to show $\hat{f}$ is smooth. Since smoothness is local, focus on $x \in \varphi(U)$. Let (V, ψ) be a smooth chart where $\varphi^{-1}(x) \in V$ and $f \circ \psi^{-1}$ is smooth. On $\varphi(U \cap V)$,



we have

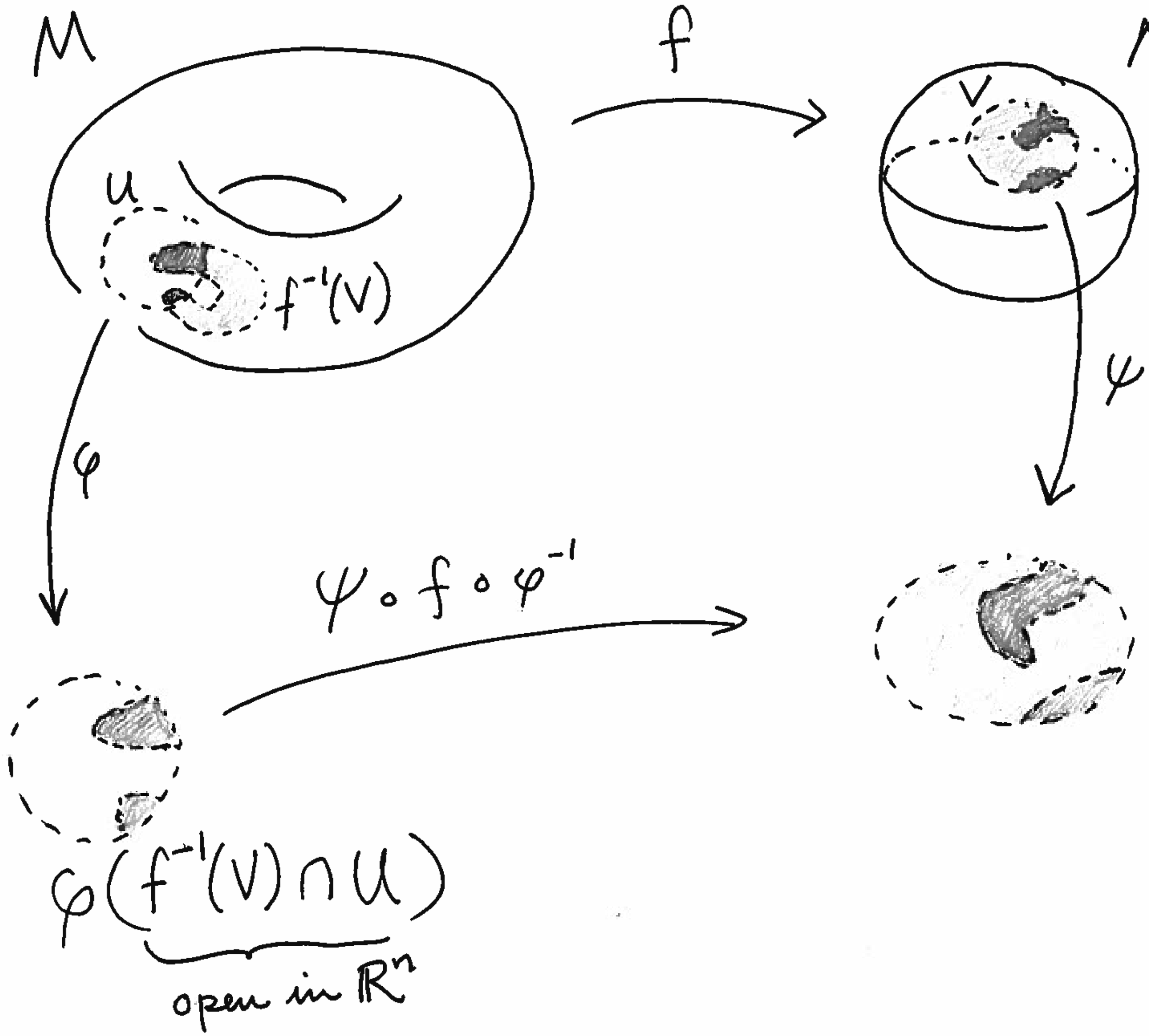
$$f \circ \varphi^{-1} = \underbrace{(f \circ \psi^{-1})}_{\text{smooth}} \circ \underbrace{(\psi \circ \varphi^{-1})}_{\text{smooth}}$$

So $\hat{f}$ is smooth. ∎

<u>Def</u>: A continuous $f : M \to N$ between smooth mflds is <u>smooth</u> if for all smooth charts (U, φ) of M and (V, ψ) of N the fn:

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \longrightarrow \psi(V)$$

is smooth.



<u>Note</u>: If $X \subseteq \mathbb{R}^n$ is <u>any</u> set, we say $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth if $\exists$ open $U \supseteq X$ and a smooth fn $\bar{f} : U \to \mathbb{R}^m$ where $\bar{f}|_X = f$.

Need this for manifolds with boundary.

Lemma [Lee pgs 34-36] Equivalently, a function $f: M \to N$ is smooth if $\forall p \in M$ there are charts $(\mathcal{U}, \varphi)$ of M and (V, ψ) of N so that ① $p \in \mathcal{U}$ and $f(\mathcal{U}) \subseteq V$
② $\psi \circ f \circ \varphi^{-1} : \varphi(\mathcal{U}) \to \psi(V)$ is smooth.

[In this version, f is <u>not</u> assumed to be continuous]

Just mention, don't write down.

Def: A <u>diffeomorphism</u> between smooth manifolds M and N is a bijection $f: M \to N$ where f and f^{-1} are both smooth.

Fact: There are 28 smooth 7-mflds $M_1, \dots, M_{27}$ so no pair are diffeomorphic but each is homeomorphic to S^7.

Ex: $N_1 = \mathbb{R}$ with $\mathcal{A}_1 = \left\{ (\mathcal{U}, \varphi) \;\middle|\; \begin{matrix} \mathcal{U} = \mathbb{R} \\ \varphi = \mathrm{id} \end{matrix} \right\}$

$N_2 = \mathbb{R}$ with $\mathcal{A}_2 = \left\{ (V, \psi) \;\middle|\; \begin{matrix} V = \mathbb{R} \\ \psi = x^{\frac{1}{3}} \end{matrix} \right\}$

$\mathcal{A}_1 \neq \mathcal{A}_2$ since $f: \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto x^{1/3}$

is <u>not</u> smooth w.r.t. $\mathcal{A}_1$ but is smooth w.r.t. $\mathcal{A}_2$.

$f: x \mapsto x^{1/3}$

$N_1 = \mathbb{R}$

φ $\quad x \mapsto x$ $\qquad\qquad$ ψ $\quad x \mapsto x^{1/3}$

$f \circ \varphi^{-1}: x \longmapsto x^{1/3}$

$f \circ \psi^{-1} = x \longmapsto (x^3)^{1/3} = x.$

But, N_1 and N_2 are diffeomorphic, via

$h: N_1 \longrightarrow N_2$. [Check this!]

$x \longmapsto x^3$

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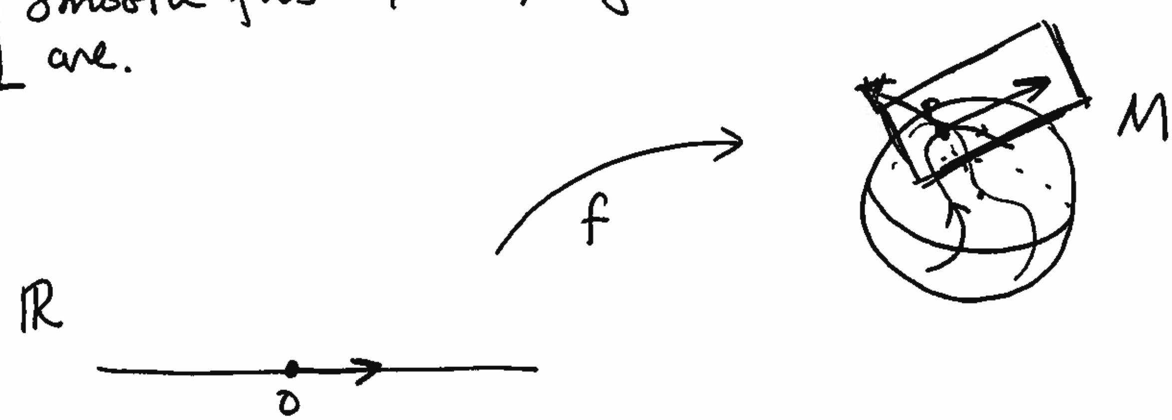
[Remember goal: Do calculus. Somehow, I've manged to define smooth fns w/o saying what there derivatives are.]



What is the derivative of f at $t=0$

Think back: $f: \mathbb{R}^n \to \mathbb{R}^k$ smooth. For $p \in \mathbb{R}^n$

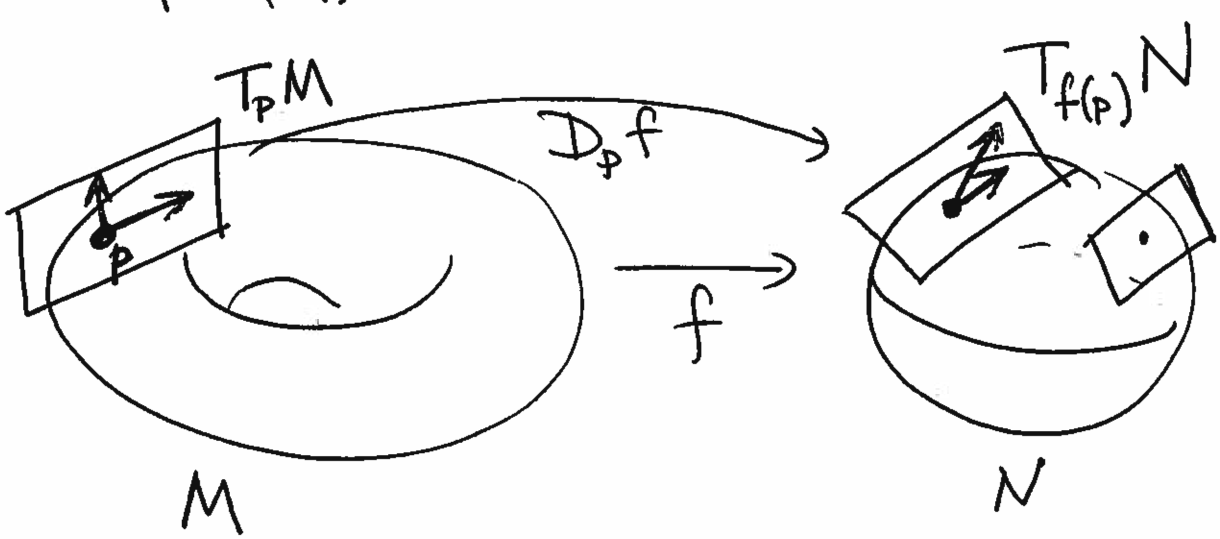
$D_p f$ is the linear trans $\mathbb{R}^n \to \mathbb{R}^k$ best approx f near p, that is

$$f(p+v) = f(p) + (D_p f)\cdot v + O(|v|^2)$$

Concretely

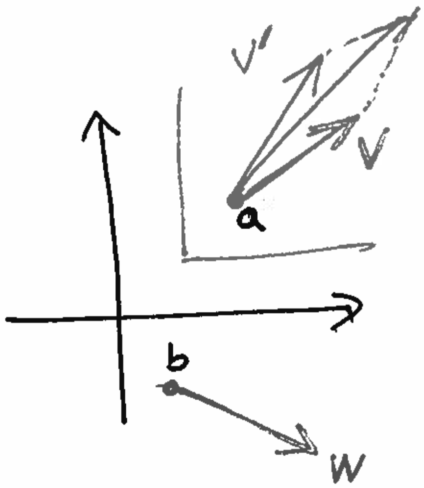
$$D_p f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

where $f = (f_1, \ldots, f_k)$. <u>Basic idea for mflds:</u>



~~[illegible]~~ First need to figure out what T_pM is.

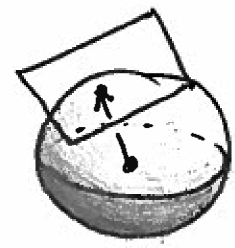
__Ex:__ $T_a\mathbb{R}^n = \{a\}\times\mathbb{R}^n = \{(a,v) \mid v\in\mathbb{R}^n\}$



$(a,v)+(a,v') = (a, v+v')$

$(a,v)+(b,w) =$ Nothing in particular

Now, view $D_pf : T_p\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}^k$



__Ex:__ $S^2 = \{x\in\mathbb{R}^3 \mid |x|=1\}$

$$T_pS^2 = \{(p,v)\in T_p\mathbb{R}^3 \mid v\cdot p = 0\}$$

__What to do in general?__ At least 3 ways to do this...

Suppose $f:(U\subseteq\mathbb{R}^n)\longrightarrow\mathbb{R}$ $\quad a\in U$

Recall

directional derivative of f at a in direction u $= \left.\dfrac{d}{dt}\big(f(a+tv)\big)\right|_{t=0} = (D_af)\cdot v$

$$D_af = \nabla f(a)$$

Let $C^\infty(\mathbb{R}^n)$ denote the set of smooth fns $\mathbb{R}^n\to\mathbb{R}$. Consider

$$D_{(a,v)} : C^\infty(\mathbb{R}^n)\longrightarrow\mathbb{R}$$

Note: (a) $D_{(a,v)}(cf+g) = c\,D_{(a,v)}f + D_{(a,v)}g$

for all $f, g \in C^\infty(\mathbb{R}^n)$, $c \in \mathbb{R}$

(b) $D_{(a,v)}(f \cdot g) = f(a)\,D_{(a,v)}g + g(a)\,D_{(a,v)}f$

A map $w : C^\infty(\mathbb{R}) \to \mathbb{R}$ is called a derivation at a if (a) w is $\mathbb{R}$-linear

(b) $w(f \cdot g) = f(a)\,w(g) + g(a)\,w(f)$

$\mathcal{D}_a = \{\text{set of all derivations at } a\} \leftarrow$ Vector space$/\mathbb{R}$.

<u>Prop</u>:

$$\begin{aligned} T_a\mathbb{R}^n &\longrightarrow \mathcal{D}_a \\ (a,v) &\longmapsto D_{(a,v)} \end{aligned}$$

is an isomorphism of vector spaces.

<u>Tricky bit</u>: That this is onto.