# Lecture 38:

For a smooth $M$, we defined

$$H^k(M) = \frac{\{\text{closed } k\text{-forms, i.e. } \omega \in \Omega^k(M) \text{ with } d\omega = 0\}}{\{\text{exact } k\text{-forms, i.e. } d\eta \text{ for } \eta \in \Omega^{k-1}(M)\}}$$

Thm: If $M$ and $N$ are homotopy equivalent, then $H^*(M) \cong H^*(N)$ as $\mathbb{R}$-algebras.

Cor: $H^k(\mathbb{R}^n) = H^k(pt) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$

Thm: Suppose $M^n$ is compact oriented and without boundary. Then $H^n(M) \twoheadrightarrow \mathbb{R}$ via the linear trans $[\omega] \longmapsto \int_M \omega$

Thm: $H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

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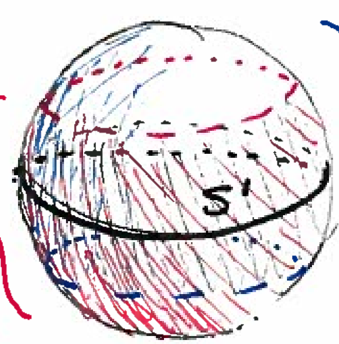
Thm: $H^k(S^2) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$

Starting point:

$S^2 = U \cup V$




$U = \{x \in S^2 \mid x_3 > -1/2\}$

$V = \{x \in S^2 \mid x_3 < 1/2\}$

Notice that $U$ and $V$ are diffeo to $\mathbb{R}^2$.

Now $U \cap V$ is diffeo to $\mathbb{R}^2 \setminus \{0\}$ which is homotopy equivalent to $S^1$ (saw last time). So know $H^*$ for $U$, $V$, and $U \cap V$; need to determine $H^*(U \cup V)$. [Will describe a general method for this but lets first do this concrete case...]

Pf: Know $k=0$ since $S^2$ is connected. For $k=1$, suppose $\omega \in \Omega^1(S^1)$ is closed. Since $H^1(U) = H^1(V) = 0$, there exists $f \in C^\infty(U)$ and $g \in C^\infty(V)$ so that

$$\omega|_U = df \quad \text{and} \quad \omega|_V = dg$$

Now on $U \cap V$ have $d(f-g) = 0$. Therefore, $f - g = \text{const } c$ on $U \cap V$. [Do aside to emph that we're using connectedness of $U \cap V$.]

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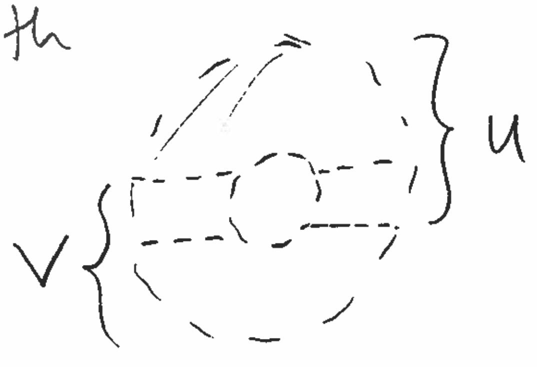
Aside: What goes wrong with




where $U \cong V \cong \mathbb{R}^2$ but $H^1(U \cup V) \neq 0$?

A: $U \cap V$ is disconnected.

Define $h = \begin{cases} f & \text{on } U \\ g - c & \text{on } V \end{cases}$ which is a <u>smooth</u> fn on $S^2$,

since $g - c = f$ on $U \cap V$. Then $dh = \omega$ and

so $H^1(S^2) = 0$.

For $k = 2$, suppose $\omega \in \Omega^2(M)$ with $\int_M \omega = 0$.

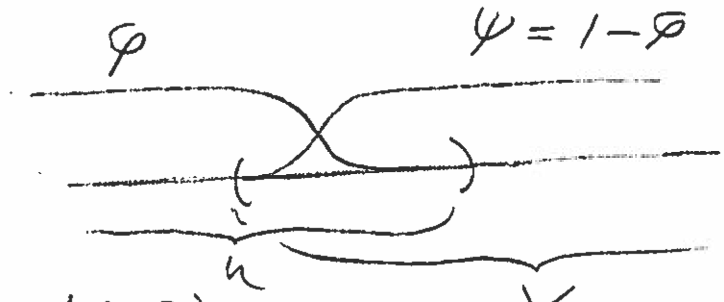
To prove the thm, enough to show that $\omega$ is exact.

Choose $\alpha \in \Omega^1(U)$ with $\omega|_U = d\alpha$ and $\beta \in \Omega^1(V)$ with $\omega|_V = d\beta$. Choose $\varphi, \psi \in C^\infty(S^2)$ with

ⓐ $\overline{\operatorname{supp} \varphi} \subseteq U$ and $\overline{\operatorname{supp} \psi} \subseteq V$.

ⓑ $\varphi = 1$ on $U \setminus V$ and $\psi = 1$ on $V - U$.

ⓒ $\varphi + \psi = 1$.




Consider

$\eta = \varphi\alpha + \psi\beta \in \Omega^1(S^2)$, and

note that $\eta = \alpha$ on $U \setminus V$ and $\eta = \beta$ on $V \setminus U$.

Hence $d\eta = \omega$ outside $U \cap V$ and there we calculate

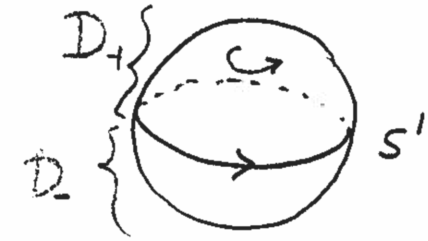
$$d(\varphi\alpha + \psi\beta) = d\varphi \wedge \alpha + d\psi \wedge \beta + \varphi\, d\alpha + \psi\, d\beta$$

$$= \underbrace{d\varphi \wedge (\alpha - \beta)}_{\text{makes sense on } S^2 \text{ as } \overline{\operatorname{supp} d\varphi} \subseteq U \cap V} + \omega$$

So, if we can find $\zeta \in \Omega^1(S^2)$ with $d\zeta = d\varphi \wedge (\alpha - \beta)$

then will have $d(\eta - \zeta) = \omega$ as desired.

On $U \cap V$, $\alpha - \beta$ is closed but is it exact?




Yes exactly when $\int_{S^1} \alpha - \beta = 0$.

Now $\int_{S^1} \alpha = \int_{\partial D_+} \alpha \overset{\text{Stokes!}}{=} \int_{D_+} \omega$ and

$\int_{S^1} -\beta = \int_{\partial D_-} \beta \underset{\text{Stokes again!}}{=} \int_{D_-} \omega$ and so $\int_{S^1} \alpha - \beta = \int_{D_+} \omega + \int_{D_-} \omega$

$$= \int_{S^2} \omega = 0.$$

Hence $\exists f \in C^\infty(U \cap V)$ with $df = \alpha - \beta$.

Now define $\zeta = -f\, d\varphi \in \Omega^1(S^2)$, and note $d\zeta = 0$ outside $U \cap V$ and on $U \cap V$ one has $d\zeta = -df \wedge d\varphi$

$= d\varphi \wedge df = d\varphi \wedge (\alpha - \beta)$ as required. ▨

[Really need a systematic way to do this... ]

Suppose we have a sequence of vector spaces

$$\cdots \xleftarrow{d_{i+1}} A_{i+1} \xleftarrow{d_i} A_i \xleftarrow{d_{i-1}} A_{i-1} \xleftarrow{d_{i-2}} \cdots$$

and linear transformations. (E.g. $A_i = \Omega^i(M)$, $d_i$ = exterior derivative)

This sequence is <u>exact</u> if $\ker d_i = \operatorname{im} d_{i-1}$ for all $i$.

<u>Mayer-Vietoris:</u> $M$ smooth, union of open $U, V$. There is a long exact sequence

$$\leftarrow H^i(U \cap V) \xleftarrow{j_U^* - j_V^*} H^i(U) \oplus H^i(V) \xleftarrow{i_U^* \oplus i_V^*} H^i(M) \leftarrow$$

$$H^{i-1}(U \cap V) \leftarrow H^{i-1}(U) \oplus H^{i-1}(V) \leftarrow H^{i-1}(M) \leftarrow$$

$$H^{i-2}(U \cap V) \leftarrow \cdots$$