Lecture 42: Poincaré Duality

So far, defined de Rham cohomology, and computed it in some examples. [S^n, surfaces, ...]

<u>Poincaré Duality</u>: Suppose M is a compact oriented n manifold without boundary.
Then $H^k(M) \cong H^{n-k}(M)$ for $0 \le k \le n$.

<u>Ex</u>. $H^n(M) \cong H^0(M) \cong \mathbb{R}$.

<u>Ex</u>: $H^*(S^1 \times S^2) =$

$*$	H
3	$\mathbb{R}$
2	$\mathbb{R}$
1	$\mathbb{R}$
0	$\mathbb{R}$

<u>Thm</u>: M as above. Then
$$H^k(M) \times H^{n-k}(M) \xrightarrow{B} \mathbb{R}$$
$$([\alpha], [\beta]) \longmapsto \int_M \alpha \wedge \beta$$
is a nondegenerate bilinear form. [Query: What does non-degenerate mean?]

<u>Notes</u>: ⓐ Well-defined since by HW have that $[\alpha] \wedge [\beta] \in H^n(M)$ can be defined as $[\alpha \wedge \beta]$ and $\int_M \omega$ depends only on $[\omega] \in H^n(M)$ by Stokes' thm.
Also, clearly bilinear.

(b) By HW, $H^k(M)$ and $H^{n-k}(M)$ are finite dim'l.

As B is non-degenerate, have $H^k(M) \hookrightarrow (H^{n-k}(M))^*$ and $H^{n-k}(M) \hookrightarrow (H^k(M))^*$. Thus $H^k(M) \cong H^{n-k}(M)$.

(c) This version is interesting even for surfaces.

$T =$  $= S^1 \times S^1$ $\quad H^1(T) = \mathbb{R}^2$ with basis $[d\theta_1]$ and $[d\theta_2]$

$$B([d\theta_1], [d\theta_1]) = 0$$
$$B([d\theta_1], [d\theta_2]) = 4\pi^2.$$

(d) $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$. Then $H^k(T^n)$ has

basis $[d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}]$ where $1 \le i_1 < \cdots < i_k \le n$.

In particular $\dim H^k(T^n) = \binom{n}{k} = \binom{n}{n-k} = \dim H^{n-k}(T^n)$.

Where does Poincaré Duality come from? Initially this seems like an indivisible statement... A closed mfld can't be divided into smaller submflds (though do have connect sum...) and it fails for $\mathbb{R}^n$: $H^n(\mathbb{R}^n) \not\cong H^0(\mathbb{R}^n)$.

<u>Thm</u>: M an oriented n-mfld. Then

$$\begin{array}{ccccc} H^k(M) & \times & H_c^{n-k}(M) & \longrightarrow & \mathbb{R} \\ [\alpha] & & [\beta] & \longmapsto & \int_M \alpha \wedge \beta \end{array}$$

compactly supported.

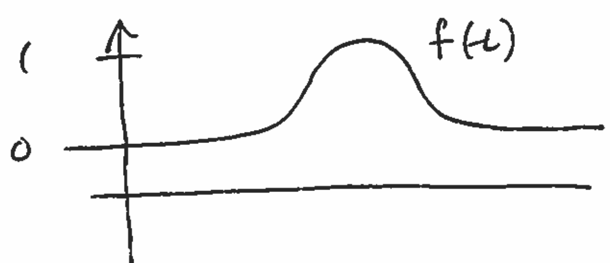
is non-degenerate.

<u>Note</u>: When M is compact, $H_c^{n-k}(M) = H^{n-k}(M)$ so this is a generalization of the earlier theorem.

Can check directly that $H_c^i(\mathbb{R}^n) = \begin{cases} \mathbb{R} & i = n \\ 0 & \text{otherwise} \end{cases}$

You did $i=0$ on HW. Here's a sketch for $H_c^1(\mathbb{R})$.

Take $\omega \in \Omega_c^1(\mathbb{R})$, say $\omega = f(t)\,dt$.



$\exists\, g \in C^\infty(\mathbb{R})$ with $dg = \omega$ but any such g is

<u>not</u> compactly supported; more precisely

$$g(t) = \int_{-\infty}^{t} f(s)\,ds$$

Alternatively, again have $H^n_c(\mathbb{R}^n) \twoheadrightarrow \mathbb{R}$ given by $[\omega] \longmapsto \int_{\mathbb{R}^n} \omega$.

One proof of Poincaré Duality is inductive, starting from the case of $\mathbb{R}^n$ and using M-V (which also exists for H^*_c). See Bott & Tu for details.

Suppose $S \subseteq M$ is a closed oriented submanifold of dim k in a closed n-manifold M. Then get a homomor: $H^k(M) \longrightarrow \mathbb{R}$.

$$[\alpha] \longmapsto \int_S \alpha$$

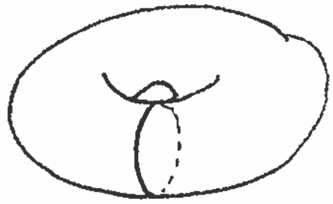
Since $(H^k(M))^* \cong H^{n-k}(M)$ under B, there must exist $[\beta] \in H^{n-k}(M)$ so that

$$\int_M [\alpha] \wedge [\beta] = \int_S [\alpha] \text{ for all}$$

$[\alpha] \in H^k(M)$. The class $[\beta]$ is called the Poincaré dual of S.

[Secretly, Poincaré Duality is a relationship between cohomology and homology...]

Ex: $T = S^1 \times S^1 \qquad S = S^1 \times \{1\}$



S

1st guess: $\beta = [d\theta_1]$

$$\int_M [d\theta_1] \wedge [d\theta_1] = 0 \neq \int_S [d\theta_1] = 2\pi$$

2nd guess: $[\beta] = \frac{1}{2\pi}[d\theta_2]$. This works since

and $[\omega] = a[d\theta_1] + b[d\theta_2]$ and so

$$\int_M [\omega] \wedge [\beta] = \int_M a[d\theta_1] \wedge [\beta] = 2\pi a$$

and $\int_S \omega = \int_S a\, d\theta_1 = 2\pi a$ ✓

Thm. Suppose S and S' are closed orient submflds of closed orient M^n of dim k and $n-k$.

Then if S and S' are transverse,

$$\#(S \cap S') = \int_M [\beta_S] \wedge [\beta_{S'}]$$

with signs