Lecture 39: Mayer-Vietoris sequence.

Computed $H^k(S^2) = \begin{cases} \mathbb{R} & \text{for } k=0,2 \\ 0 & \text{otherwise} \end{cases}$ using $S^2 = U \cup V$

where U, V are open sets where we knew H^* of U, V and $U \cap V$.

Today: General algebraic tool for this.

Exact sequence:

linear transformation

vector spaces

[More gen: groups and homomorphisms.]

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \xrightarrow{\alpha_{n+1}} \cdots$$

where $\ker \alpha_n = \operatorname{im} \alpha_{n-1}$ for all n.

Note: Must have $\alpha_n \circ \alpha_{n-1} = 0$ but exactness is more than this. [Morally the "cohomology" of (A_*, α_*) is 0.]

Ex: $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact $\iff$ α is 1-1

$B \xrightarrow[\beta]{} C \longrightarrow 0$ is exact $\iff$ α is onto.

So exactness of "Short exact sequence"

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

means that B has a subspace A with quotient C.

In particular for vector spaces this means

$$B \cong A \oplus C.$$

Mayer–Vietoris: M smooth, union of open U, V.

There is an exact sequence

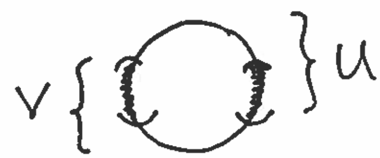
$$\to H^k(M) \xrightarrow{i_U^* \oplus i_V^*} H^k(U) \oplus H^k(V) \xrightarrow{j_U^* - j_V^*} H^k(U \cap V)$$

$$\xrightarrow{\delta} H^{k+1}(M) \longrightarrow H^{k+1}(U) \oplus H^{k+1}(V) \longrightarrow H^{k+1}(U \cap V)$$

$$\xrightarrow{\delta} H^{k+2}(M) \longrightarrow$$

where $i_U : U \hookrightarrow M$, $i_V : V \hookrightarrow M$, $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$

Ex: $M = S^1$



$$0 \to \underset{\mathbb{R}}{H^0(M)} \xrightarrow{i} \underset{\mathbb{R} \oplus \mathbb{R}}{H^0(U) \oplus H^0(V)} \xrightarrow{j} \underset{\mathbb{R}^2}{H^0(U \cap V)} \xrightarrow{\delta} H^1(M) \to \underset{0}{H^1(U)} \oplus \underset{0}{H^1(V)} \to$$

$$\dim(\operatorname{im} i) = 1 \Rightarrow \dim(\ker j) = 1 \Rightarrow \dim(\operatorname{im} j) = 1$$

$$\Rightarrow \dim(\ker \delta) = 1 \Rightarrow \dim(\underset{\underset{H^1(M)}{\parallel}}{\operatorname{im} \delta}) = 1$$

So $H^1(M) = \mathbb{R}$.

③

<u>Ex:</u> $M = S^2$.

$$0 \to \overset{\mathbb{R}}{H^0(M)} \xrightarrow{i} \overset{\mathbb{R}^2}{H^0(U)\oplus H^0(V)} \xrightarrow{j} \overset{\mathbb{R}}{H^0(U\cap V)} \longrightarrow \quad 0$$

$$\to \underset{\boxed{0}}{H^1(M)} \to \underset{0}{H^1(U)\oplus H^1(V)} \to \underset{\mathbb{R}}{H^1(U\cap V)} \longrightarrow$$

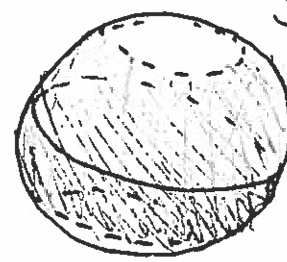
$$\to \underset{\boxed{\mathbb{R}}\,\checkmark}{H^2(M)} \to \underset{0}{H^2(U)\oplus H^2(V)} \longrightarrow \cdots$$

im j has dim 1
$\Rightarrow$ j onto
$\Rightarrow$ Ker δ = everything
$\Rightarrow$ im $\delta = 0$

<u>Ex:</u> $S^n = U \cup V$ where $U \cong V \cong \mathbb{R}^n$ and $U \cap V$ is homeotopy equiv to S^{n-1}

Inductively use M-V to prove

S^n



S^{n-1}

<u>Thm</u>: $H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, n \\ 0 & \text{otherwise} \end{cases}$

$U = \{x_{n+1} > -1/2\}$

$V = \{x_{n+1} < +1/2\}$

Proof of M-V uses "homological algebra".

$$\cdots \longrightarrow \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \xrightarrow{d_{k+1}} \Omega^{k+2}(M) \longrightarrow \cdots$$

is called a "cochain complex." Abstractly, this is just

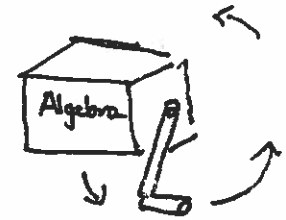
$$\cdots \longrightarrow A^k \xrightarrow{\alpha_k} A^{k+1} \xrightarrow{\alpha_{k+1}} A^{k+2} \longrightarrow \cdots \quad \text{with } \alpha_{k+1}\circ\alpha_k = 0$$

Have

$$0 \longrightarrow \Omega^*(M) \xrightarrow{i_U^* \oplus i_V^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V) \longrightarrow 0$$

cochain maps

which is exact. (partition of unity argument).



$\rightsquigarrow$ long exact sequence in cohomology.

Suppose $F: S^n \to S^n$ is smooth. The <u>degree</u> of F is the number $\deg(f) \in \mathbb{R}$ so that

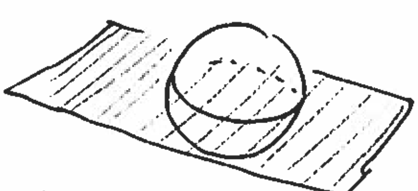
$$F^*([\omega]) = \deg(f)\,[\omega] \quad \text{for all } [\omega] \in H^n(S^n) = \mathbb{R}$$

Equivalently, for any $\omega \in \Omega^n(S^n)$ we have

$$\int_{S^n} F^*\omega = \deg(f) \int_{S^n} \omega$$

<u>Ex:</u> $\deg(id_{S^n}) = 1$

$\deg(\text{Reflect in } \mathbb{R}^n) = -1$



$(x_0, \ldots, x_n) \longmapsto (x_0, \ldots, x_{n-1}, -x_n)$

$\deg\left(\begin{smallmatrix}\text{const}\\ \text{map}\end{smallmatrix}\right) = 0$

$F^*\omega = 0$ $\qquad$ ω

since if ω is the standard volume form on S^n, this sends ω to $-\omega$.

<u>Ex</u>: $S^1 \xrightarrow[F]{} S^1$ $d\theta$ gen $H^1(S^1)$

$z \longrightarrow z^2$ $F^*(d\theta) = 2\,d\theta$ $\deg F = 2$

<u>Ex</u>: $S^1 \xrightarrow[F]{} S^1$ $\deg F = n$.

$z \longmapsto z^n$

<u>Thm</u>: For any $F: S^n \longrightarrow S^n$, $\deg(f) \in \mathbb{Z}$

For any regular value $q \in S^n$ we have

$$\deg F = \sum_{p \in F^{-1}(q)} \begin{cases} +1 & \text{if } dF_p \text{ is orient pres.} \\ -1 & \text{if } dF_p \text{ is orient reversing.} \end{cases}$$

<u>Note</u>: Homotopic maps must have the same degree.

In fact, $F, G: S^n \longrightarrow S^n$ are homotopic iff $\deg F = \deg G$.