Lecture 41: Applications

Degree of smooth $F: S^n \to S^n$ defined by $F^*[\omega] = (\deg F)[\omega]$ for all $[\omega] \in H^n(S^n)$. Always in $\mathbb{Z}$.

Thm. S^n has a nowhere vanishing vector field iff n is odd.

If M and N are compact oriented n-mflds without ∂, then can define

$$\deg(F: M \to N) = \int_M F^*\omega \Big/ \int_N \omega$$

where $[\omega] \neq 0$ in $H^n(N)$. Well-defined once know $H^n(M) \cong H^n(N) \cong \mathbb{R}$. [Still in $\mathbb{Z}$, etc.]

Fact: $F, G: S^n \to S^n$ are homotopic if and only if $\deg F = \deg G$.

[Another classic application of deg is the F.T.A.]

Thm: When n is even, the only group that can act freely on S^n is $\mathbb{Z}/2\mathbb{Z}$.

[In contrast S^1 acts on any odd-dim'l sphere.]

means that $g \cdot x = x \Rightarrow g = id$.

<u>Proof</u>: Since $\deg(F\circ G) = (\deg F)(\deg G)$,

the degree of any of any diffeomorphism is ± 1.

So get a homomorphism $G \longrightarrow \{\pm 1\}$. If $g \neq 1$

$g \longmapsto \deg(\Theta_g)$

in G, then Θ_g has no fixed points $\underset{\text{last time}}{\Longrightarrow}$ $\deg \Theta_g = (-1)^{n+1}$

$= -1$

So $G \to \{\pm 1\}$ has no kernel and so

$G \cong \mathbb{Z}/2\mathbb{Z}$. ▨

<u>Brouwer Fixed Point Thm</u>: Let $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Any continuous $F: D^n \longrightarrow D^n$ has a fixed pt,

i.e. $\exists x \in D^n$ with $F(x) = x$.

[Demo.]

<u>Lemma</u>: There does not exist a smooth $R: D^n \longrightarrow \partial D^n = S^{n-1}$

which is the identity on ∂D^n.

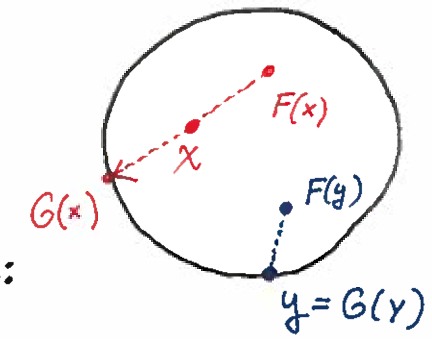
<u>Pf</u>: By HW, any retract induces an injection $H^*(\partial D^n) \to H^*(D^n)$.

But $H^{n-1}(\partial D^n) = \mathbb{R}$ and $H^{n-1}(D^n) = 0$ when $n \geq 1$

and $H^0(\partial D^1) = \mathbb{R}^2$ and $H^0(D^1) = \mathbb{R}$. So no such R exists. ▨

Lemma: Any smooth $F: D^n \to D^n$ has a fixed point.

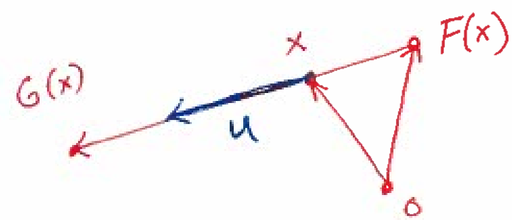
Proof: Suppose F has no fixed pts.



Define $G: D^n \to \partial D^n$ by this picture:

Note that G is the ident on ∂D^n. If G is smooth, can apply last lemma to get a contradiction.

Set $u = \dfrac{x - F(x)}{|x - F(x)|}$



and $\lambda = -x \cdot u + \sqrt{\underbrace{1 - |x|^2 + (x \cdot u)^2}_{\star}}$. Claims:

① $\lambda \geq 0$ since $1 - |x|^2 \geq 0$ and so $\sqrt{\star} \geq |x \cdot u|$.

② $G(x) = x + \lambda u$ since $(x + \lambda u) \cdot (x + \lambda u) = \text{unpleasantness} = 1$

③ $\star > 0$: If it were 0, must have $|x| = 1$ and u at right angle to x, which is silly.

So since u is a smooth fn of x as is λ (by 3) we have that G is smooth as needed. ▨

Proof of B.F.T.

(a) Take Math 525 next Spring.

[Developes (Co)homology for arbitrary topological spaces.]

(b) Given $F : D^n \to D^n$ continuous and without fixed pts, choose $\varepsilon > 0$ so that $|F(x) - x| > \varepsilon$ for all $x \in D^n$.

Use e.g. Weierstraß approximation to find a smooth $F_1 : D^n \to D^n$ where $|F(x) - F_1(x)| < \varepsilon/2$ on D^n.

Then F_1 has no fixed points since if $F_1(x) = x$ we have

$$|F(x) - x| = |F(x) - F_1(x)| < \varepsilon/2.$$

So done by the smooth case.

Q.E.D.