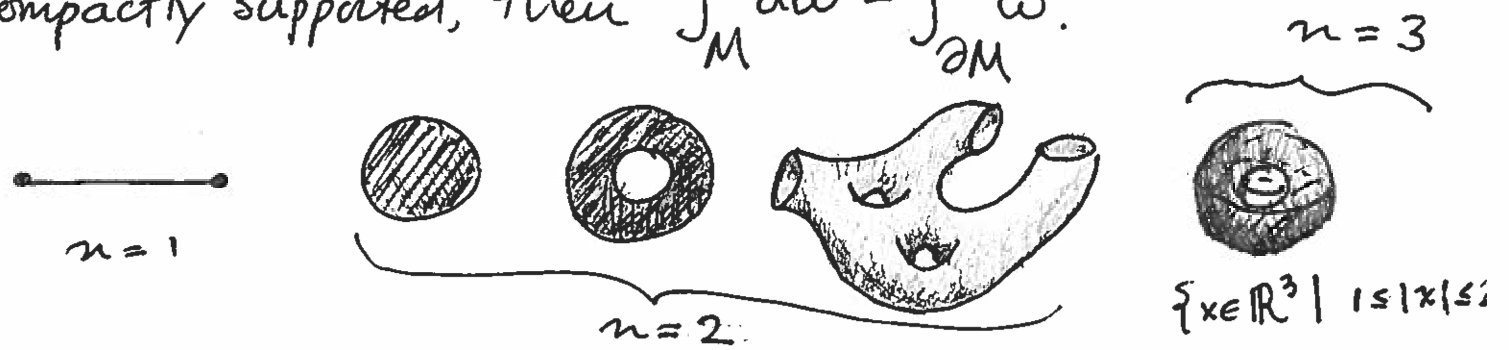
Lecture 33: Proof of Stokes Theorem

<u>Stokes Thm:</u> Suppose that M is a smooth oriented n-manifold with boundary. If $\omega \in \Omega^{n-1}(M)$ is compactly supported, then $\int_M d\omega = \int_{\partial M} \omega$.



<u>Pf:</u> Let $(\underbrace{U_i, \varphi_i}_{\text{chart}}, \underbrace{\psi_i}_{\text{bump fn.}})$ be a partition of unity for M

Define $\omega_i = \psi_i \omega$. Since ω is compactly supported, only finitely many ω_i are non-zero.

[This is just local finiteness; only finitely many $\operatorname{supp} \psi_i$ can meet $\overline{\operatorname{supp} \omega}$.]

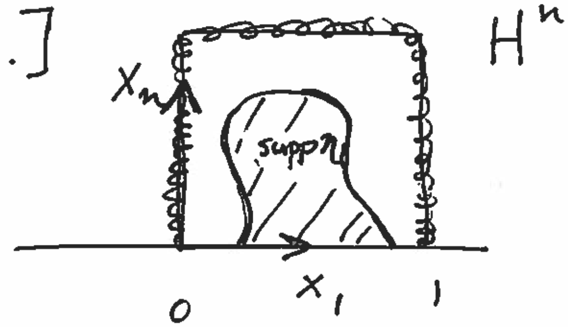
Say $\omega = \sum_{i=1}^{N} \omega_i$. Enough to show $\int_M d\omega_i = \int_{\partial M} \omega_i$ for each i, since then $\int_M d\omega = \int_M d(\Sigma \omega_i) =$

$$\int_M \Sigma\, d\omega_i = \Sigma \int_M d\omega_i = \Sigma \int_{\partial M} \omega_i = \int_{\partial M} \Sigma\, \omega_i = \int_{\partial M} \omega.$$

Since $\overline{\text{supp}\,\omega_i} \subseteq \mathcal{U}_i$, can compute $\int_M d\omega_i$ and $\int_{\partial M} \omega_i$ via the single chart $\varphi_i : \mathcal{U}_i \to \mathbb{R}^n$ or H^n

So we have reduced to the case $M = H^n$, and by changing coor we can assume that $\text{supp}\,\omega \subseteq [0,1]^n$.

Now $\omega = \sum f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and again enough to prove it for each term. Two cases [up to changing coordinates.]



Ⓐ $\eta = f(x)\, dx_2 \wedge \cdots \wedge dx_n$

Now $\int_{\partial M} \eta = 0$ since $\eta = 0$ when all input vectors are in $T_p \partial M$. Have $d\eta = \frac{\partial f}{\partial x_1}(x)\, dx_1 \wedge \cdots \wedge dx_n$

So $\int_M d\eta = \int_0^1 \cdots \int_0^1 \frac{\partial f}{\partial x_1}\, dx_1\, dx_2 \cdots dx_n$

$$= \int_0^1 \cdots \int_0^1 \underset{0}{f(1, x_2, \ldots, x_n)} - \underset{0}{f(0, x_2, \ldots, x_n)}\, dx_2 \cdots dx_n$$

$= 0$ as needed.

(B) $\eta = f(x)\, dx_1 \wedge \cdots \wedge dx_{n-1}$

$$\int_{\partial H^n} \eta = (-1)^n \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_{n-1}, 0)\; dx_1 \cdots dx_{n-1}$$

because orient on ∂H^n isn't always the same as that on $\mathbb{R}^{n-1}$!

$d\eta = (-1)^{n-1} \dfrac{\partial f}{\partial x_n}(x)\, dx_1 \wedge \cdots \wedge dx_n$. So

$$\int_{H^n} d\omega = \int_0^1 \cdots \int_0^1 (-1)^{n-1} \frac{\partial f}{\partial x_n}(x)\; dx_n\, dx_1 \cdots dx_{n-1}$$

$$= (-1)^{n-1} \int_0^1 \cdots \int_0^1 \overbrace{f(x_1, \dots, x_{n-1}, 1)}^{0} - f(x_1, \dots, x_{n-1}, 0)\; dx_1 \cdots dx_{n-1}$$

$$= (-1)^n \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_{n-1}, 0)\; dx_1 \cdots dx_{n-1}$$

$$= \int_{\partial H^n} \eta .$$

So we're done! Q.E.D.

[This proof was fairly easy but somewhat formal. To get more intuition about Stokes theorem, lets see what it says for a Riemannian metric...]

(M, g) oriented Riemannian mfld with boundary.

$\omega_g \in \Omega^n(M)$ the volume form.

[Philosophy: a R-metric allows us to go between various kinds of tensors...]

Define

$$* : C^\infty(M) \longrightarrow \Omega^n(M) \quad \text{which is an isomorphism}$$
$$f \longrightarrow f\omega_g \quad \text{[Example of Hodge star.]}$$

Note for $f \in C^\infty(M)$ we defined $\int_M f \, dVol = \int_M *f$

Also, have
$$\beta : \mathfrak{X}(M) \longrightarrow \Omega^{n-1}(M)$$
$$X \longrightarrow X \lrcorner \, \omega_g$$

Define the divergence operator $\operatorname{div} : \mathfrak{X}(M) \longrightarrow C^\infty(M)$

by $\operatorname{div} X = *^{-1}(d(\beta(X)))$

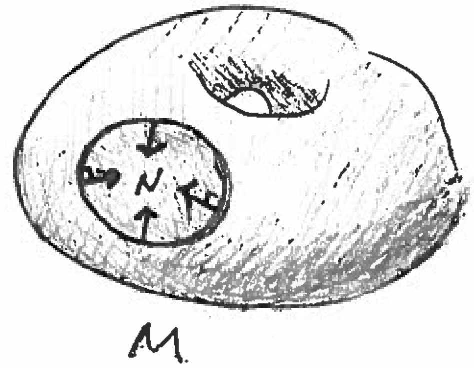
Equivalently

$$(\operatorname{div} X)\, \omega_g = d(X \lrcorner \, \omega_g)$$

<u>Divergence Theorem</u>: (M,g) oriented Riemannian manifold with boundary. If $X \in \mathfrak{X}(M)$ is compactly supported, and N is the outward-pointing unit normal vector field on ∂M, then

$$\int_M (\operatorname{div} X)\, dVol_M$$

$$= \int_{\partial M} g(N, X)\, dVol_{\partial M}$$



where the Riemannian metric on ∂M is just the restriction of the one on M.

This has a physical interpretation....