Lecture: More on cohomology.

$\omega \in \Omega^k(M)$ is closed if $d\omega = 0$; it is exact if $\exists\, \eta \in \Omega^{k-1}(M)$ with $d\eta = \omega$.

de Rham cohomology:

$$H^k(M) = \{\text{closed } k\text{-forms}\} \big/ \{\text{exact } k\text{-forms}\}$$

$$= \left\{ [\omega] \;\middle|\; \begin{array}{l} \omega \in \Omega^k(M) \text{ where } [\omega] = [\omega'] \\ \text{if } \omega - \omega' \text{ is exact.} \end{array} \right\}$$

For a smooth $F: M \longrightarrow N$, get $F^*: H^k(N) \longrightarrow H^k(M)$

Set $H^*(M) = \bigoplus_{i=0}^{n} H^i(M)$. This is an algebra via the multiplication $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$

[Well-defined proved on HW.]

Prop: If M is connected, then $H^0(M) = \mathbb{R}$.

Pf: Have $H^0(M) = \left(\begin{array}{c}\text{closed}\\ \text{0-forms}\end{array}\right)$ since no $\Omega^{-1}(M)$.

A function $f \in \Omega^0(M)$ has $df = 0$ exactly when it is locally (and hence globally) constant.

So $H^0(M) = \begin{array}{c}\text{constant}\\ \text{functions}\end{array} \cong \mathbb{R}$.

Query:

$H^0($  $) = \mathbb{R}^2$

More generally, $H^*(M \amalg N) = H^*(M) \times H^*(N)$

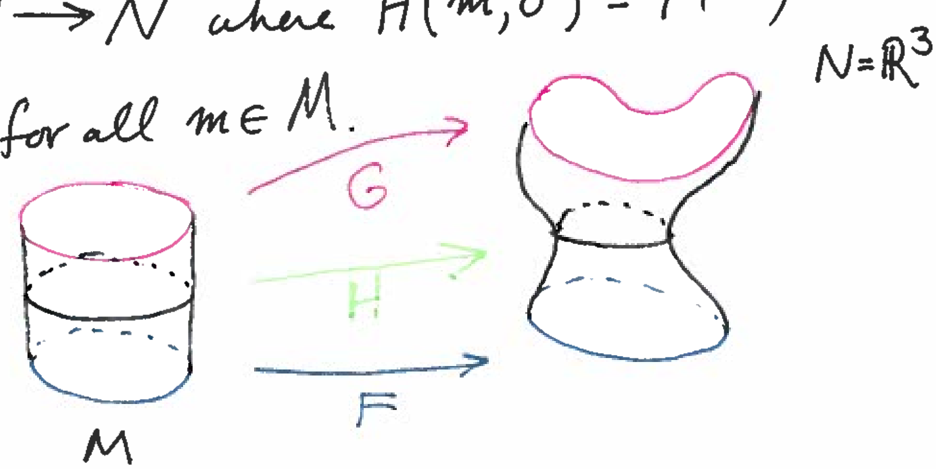
Prop: M^n cpt oriented w/o boundary. Then there is an onto homomorphism $H^n(M) \xrightarrow[\nu]{} \mathbb{R}$.

Pf: Define $\nu([\omega]) = \int_M \omega$. Well-defined since if $\omega' = \omega + d\eta$ have $\int_M \omega' = \int_M \omega + \int_M d\eta = \int_M \omega$.

Onto since if ω is an orientation form, then $\int_M \omega > 0$. ▨

Prop: $H^i(pt) = \begin{cases} \mathbb{R} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$ [Only space where we know everything. Need more tools...]

Two smooth maps $F, G : M \to N$ are homotopic if $\exists$ a smooth $H : M \times [0,1] \to N$ where $H(m,0) = F(m)$ and $H(m,1) = G(m)$ for all $m \in M$.



<u>Thm</u>: If F and $G : M \to N$ are homotopic, they induce the <u>same</u> map $H^*(N) \longrightarrow H^*(M)$.

<u>Cor</u>: $H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } * = 0 \\ 0 & \text{otherwise.} \end{cases}$

<u>Pf of Cor</u>: Define $H : \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$ by $H(x,t) = tx$.

This is a homotopy from $id_{\mathbb{R}^n}$ to $\left(\begin{matrix}\text{const map} \\ \mathbb{R}^n \longrightarrow 0\end{matrix}\right)$ $= c$. Hence for $i > 0$ have

$$\begin{tikzcd}
H^i(\mathbb{R}^n) \arrow[rr, "(id_{\mathbb{R}^n})^* = id_{H^i(\mathbb{R}^n)}", "c^*"'] \arrow[dr, "\left(\begin{smallmatrix}\text{inclusion} \\ 0 \to \mathbb{R}^n\end{smallmatrix}\right)"'] & & H^i(\mathbb{R}^n) \\
& H^i(pt) \arrow[ur, "\left(\begin{smallmatrix}\text{const} \\ \text{map } \mathbb{R}^n \to 0\end{smallmatrix}\right)"'] &
\end{tikzcd}$$

Since $H^i(pt) = 0$, must have $H^i(\mathbb{R}^n) = 0$

$\blacksquare$

To compare F^* and G^*, pick $[\omega] \in H^k(N)$. Then $F^*[\omega] = G^*[\omega]$ is the same as

$$F^*\omega - G^*\omega = d\eta \quad \text{for some } \eta \in \Omega^{k-1}(M)$$

Set $I = [0,1]$ and for $t \in I$ consider the inclusion $i_t : M \to M \times I$ given by $i_t(m) = (m,t)$

Notice that $F^*\omega = i_0^*\alpha$ and $G^*\omega = i_1^*\alpha$ where $\alpha = H^*\omega$.

<u>Homotopy Operator Lemma:</u>

Let M^n be smooth w/o boundary. There exists a linear map $h: \Omega^k(M \times I) \longrightarrow \Omega^{k-1}(M)$ for all k so that $d(h\alpha) + h(d\alpha) = \alpha_1 - \alpha_0$ ⊛ where $\alpha \in \Omega^k(M \times I)$ and $\alpha_t = i_t^*\alpha$. In particular, if α is closed then $[\alpha_1] = [\alpha_0]$ in $H^k(M)$.

<u>Query</u>: How many have seen this type of thing before?

[Explain why the last sentence follows from the first. Say why useful to have h defined on non-closed forms.]

Pf: Define

$$(h\alpha)(\underset{\in T_m M}{v_1, \dots, v_{k-1}}) = \int_0^1 \alpha_{(m,t)}\left(\frac{\partial}{\partial t}, v_1, \dots, v_{k-1}\right) dt$$

To check ⊛ at $m \in M$, choose coor $(x_1, \dots, x_n)$ near p. By linearity and permuting coordinates, enough to check for

$$\alpha = f(x,t)\, dx_1 \wedge \dots \wedge dx_k$$

and

$$\beta = f(x,t)\, dt \wedge dx_1 \wedge \dots \wedge dx_{k-1}$$

For α, note $h\alpha = 0$ and $h(d\alpha) =$

$$h\left(\frac{\partial f}{\partial t}(x,t)\, dt \wedge dx_1 \wedge \dots \wedge dx_k\right)$$

$$= \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)\right) dx_1 \wedge \dots \wedge dx_k$$

$$= \left(f(x,1) - f(x,0)\right) dx_1 \wedge \dots \wedge dx_k = \alpha_1 - \alpha_0$$

For β, work this out as HW problem #5 on assignment 12.