Lecture 37: Homotopies and Cohomology.

Smooth maps $F, G : M \to N$ are smoothly homotopic if $\exists\, H : M \times I \to N$ with $H \circ i_0 = F$ and $H \circ i_1 = G$.

Here $i_t : M \to M \times I$, $m \mapsto (m, t)$ is the inclusion at height t.

Thm. Homotopic maps induce the same map on cohomology.

Homotopy Operator Lemma Let M^n be smooth w/o boundary. There exist linear maps $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$ for all k so that

$$d(h\alpha) + h(d\alpha) = \alpha_1 - \alpha_0 \qquad (\star)$$

where $\alpha \in \Omega^k(M \times I)$ and $\alpha_t = i_t^*(\alpha)$.

In particular, if α is closed then $[\alpha_1] = [\alpha_0]$ in $H^k(M)$. This also implies the theorem by taking $\alpha = H^* \omega$ for $\omega \in \Omega^k(N)$ as then $\alpha_0 = F^* \omega$ and $\alpha_1 = G^* \omega$.

[Say why useful to have defined for non-closed forms.]

Pf: Define

$$(h\alpha)(\underbrace{v_1,\dots,v_{k-1}}_{\text{in } T_mM}) = \int_0^1 \alpha_{(m,t)}\overbrace{\left(\tfrac{\partial}{\partial t}, v_1, \dots, v_{k-1}\right)}^{\text{in } T_{(m,t)}M\times I}\, dt$$

To check ⊛ at $m \in M$, work in local coordinates $(x_1,\dots,x_n)$

By linearity of h and permuting coordinates, enough to check

$$\alpha = f(x,t)\, dx_1\wedge\dots\wedge dx_k$$

and

$$\beta = f(x,t)\, dt\wedge dx_1\wedge\dots\wedge dx_{k-1}$$

For α, note $h\alpha = 0$ and $h(d\alpha) =$

$$h\left(\tfrac{\partial f}{\partial t}(x,t)\, dt\wedge dx_1\wedge\dots\wedge dx_k\right)$$

$$= \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)\, dt\right) dx_1\wedge\dots\wedge dx_k$$

$$= \big(f(x,1) - f(x,0)\big)\, dx_1\wedge\dots\wedge dx_k = \alpha_1 - \alpha_0$$

For β, work this out as problem #5 on HW 12. ▨

[So now we know $H^*(\mathbb{R}^n)$, but still need better tools... But first a concrete example.]

Thm: $H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k=0,1 \\ 0 & \text{otherwise.} \end{cases}$

Pf. Have $\partial : H^1(S^1) \twoheadrightarrow \mathbb{R}$ given by $\partial([\omega]) = \int_{S^1} \omega$.
Suppose $[\omega] \in \ker \partial$. Use angle coordinates θ and write $\omega = f\,d\theta$ for $f \in C^\infty(S^1)$. Define $g \in C^\infty(S^1)$ by $g(\theta \in [0, 2\pi]) = \int_0^\theta f(t)\,dt$; this makes sense because $\int_0^{2\pi} f\,dt = 0$. Now $dg = \omega$ and so $[\omega] = 0$ in $H^1(S^1)$. So $H^1(S^1) = \mathbb{R}$. ▨

Called a retract of D^2 to S^1

Cor: $\nexists$ a smooth $R : \mathbb{R}^2 \to S^1$ with $R|_{S^1} = id_{S^1}$.

Pf: By HW12 #2, $R^* : H^1(S^1) \to H^1(\mathbb{R}^2)$ must be injective. But there's no such map $\mathbb{R} \to 0$. ▨

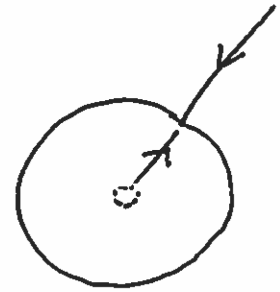
Def: Manifolds M and N are smoothly homotopy equivalent if $\exists$ smooth maps $F : M \to N$ and $G : N \to M$ with $G \circ F$ homotopic to id_M and $F \circ G$ homotopic to id_N.

Ex: • $\mathbb{R}^n$ is homotopy equivalent to a point.

• $M = \mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to $N = S^1$

$F : M \longrightarrow N$ is $F(x) = \frac{x}{|x|}$

$G : N \longrightarrow M$ inclusion



Check: $F \circ G = id_N$ ✓

$G \circ F = F$ is homotopic to id_M via $H : M \times I \to M$ given by $tx + (1-t)\frac{x}{|x|}$.

Cor: If M and N are homotopy equivalent, then $H^*(M) \cong H^*(N)$.

Pf: Consider $H^*(M) \underset{G^*}{\overset{F^*}{\rightleftarrows}} H^*(N)$. Now $F^* \circ G^*$

$= (G \circ F)^* = (id_M)^* = id_{H^*(M)}$. Similarly,

$G^* \circ F^* = (F \circ G)^* = (id_N)^* = id_{H^*(N)}$.

So F^* and G^* are inverse bijections. $\square$