Integration on Manifolds (Lecture 30)

An orientation on M^n is a locally consistent choice of orientation on each T_pM; it can be specified by $\omega \in \Omega^n(M)$ where each $\omega_p \neq 0$.

Partition of Unity: Countable (U_i, φ_i, ψ_i) where (U_i, φ_i) are smooth charts, $\psi_i : M \to [0,1]$ smooth fns and (a) $\operatorname{supp} \psi_i \subseteq U_i$ (b) Every p is in finitely many $\operatorname{supp} \psi_i$

(c) $\sum_i \psi_i = 1$.

A form $\omega \in \Omega^n(M)$ is compactly supported if $\operatorname{supp} \omega = \{p \in M \mid \omega_p \neq 0\}$ is contained in a cpt set.

Def: $\underset{\text{open}}{U \subseteq \mathbb{R}^n}$ For compactly supported $\omega \in \Omega^n(U)$, write $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ where $f \in C^\infty(\mathbb{R}^n)$ is 0 outside U. Define

$$\int_U \omega = \int_U f\, dV = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

where $\prod [a_i, b_i]$ is any cpt box containing $\operatorname{supp} f = \operatorname{supp} \omega$.

Suppose M is an oriented n-mfld and $\omega \in \Omega^n(M)$ is compactly supported. Define

$$\int_M \omega = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\psi_i \cdot \omega)$$

Really only a finite sum.

where (U_i, φ_i, ψ_i) is a partion of unity where each φ_i is orientation preserving. [Query: Why can we add the last condition?]

<u>Note</u>: Since $\operatorname{supp} \psi_i \subseteq U_i$, the form $(\varphi_i^{-1})^*(\psi_i \omega)$ is smooth and compactly supported on $\varphi_i(U_i)$.

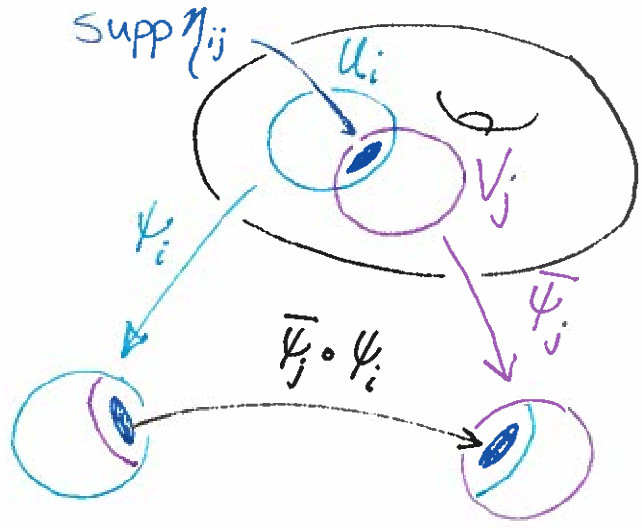
<u>Thm.</u> $\int_M \omega$ does not depend on the choice of the partition of unity.

<u>Proof</u>: First note that $\int_M \omega$ with respect to some fixed (U_i, φ_i, ψ_i) is linear in ω. Suppose $(V_j, \overline{\varphi}_j, \overline{\psi}_j)$ is another partition of 1. Consider

$$\omega = \sum_{i,j} \overbrace{\psi_i \overline{\psi}_j \omega}^{\eta_{ij}}$$

[which is really a finite sum.]

Now we have $\overline{\operatorname{supp} \eta_{ij}} \subseteq U_i$ and V_j. By linearity we just need to check $\int_M \eta_{ij}$ is the same in both partitions of unity.



To see this, note

$$(\psi_i^{-1})^*(\eta_{ij})$$

is

$$(\bar{\psi}_j \circ \psi_i)^*\left((\bar{\psi}_j^{-1})^* \eta_{ij}\right)$$

and so the thm follows from

<u>Lemma</u>: Suppose $F: U \longrightarrow V$ is an orient. pres. diffeo of open subsets of $\mathbb{R}^n$. If $\eta \in \Omega^n(V)$ is compactly supported then

$$\int_V \eta = \int_U F^* \eta$$

<u>Idea</u>: Write $\eta = f(x)\, dx_1 \wedge \cdots \wedge dx_n \quad f \in C^\infty(V)$

and $F^*\eta = g(x)\, dx_1 \wedge \cdots \wedge dx_n \quad g \in C^\infty(U)$

Have

Columns of DF

$$\begin{aligned} g(x) &= (F^*\eta)\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big|_x\right) \\ &= \eta_{F(x)}\left(dF\left(\frac{\partial}{\partial x_1}\Big|_x\right), \ldots, dF\left(\frac{\partial}{\partial x_n}\Big|_x\right)\right) \\ &= f(F(x)) \cdot \det(D_x F)^t = f(F(x)) \det(D_x F) \end{aligned}$$

Since F is orient. pres, $\det(D_x F) > 0$. So lemma is now "just" the change of variables formula for multiple integrals. ∎

Volumes of Riemannian Manifolds:

M^n oriented with Riemannian metric g.

The volume form $\omega_g \in \Omega^n(M)$ is defined by

$$\omega_g(v_1, \ldots, v_n) = \left.\begin{array}{l} \text{signed vol of the} \\ \text{pallelopiped spanned} \\ \text{by } v_1, \ldots, v_n \in T_pM \\ \text{with respect to } g_p. \end{array}\right\} \Rightarrow \omega_g = 1 \text{ on any pos. oriented orthonormal basis.}$$

More explicitly

$$\omega_g(v_1, \ldots, v_n) = \sqrt{\det(g_p(v_i, v_j))}$$

if $(v_1, \ldots, v_n)$ are a pos. oriented basis.

Reason this is the right formula: On $\mathbb{R}^n$ with g_{DOT}, given $(v_1, \ldots, v_n)$ consider the matrix $A = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix}$

Then $\det(g_{DOT}(v_i, v_j)) = \det(AA^t) = (\det A)^2$

$= (\text{sign vol span } v_i)^2.$

For $f \in C^\infty(M)$ can now define $\int_M f\, dV$

as $\int_M f\, \omega_g$.