Lecture 31: Exterior Differentiation

$$\Omega^k(M) = \left\{ \begin{array}{c} \text{smooth } k\text{-forms} \\ p \longmapsto \Lambda^k(T_pM) \end{array} \right\}$$

Goal: Define $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

generalizing $\begin{array}{c} C^\infty(M) \longrightarrow \Omega^1(M). \\ f \longmapsto df \end{array}$ Note that

by definition $\Lambda^0 V = \mathbb{R}$, so $C^\infty(M) = \Omega^0(M)$.

Exterior Algebra: $\Lambda(V) = \bigoplus_{k=0}^{\dim V} \Lambda^k V$

Opperations are addition and $\wedge$ product; which make it an associative graded (anti) commutative algebra: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \alpha \in \Lambda^k V,\ \beta \in \Lambda^l V$

Algebra of differential forms: $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$

Thm: M a smooth mfld, poss. with boundary. There are unique maps $\underbrace{d : \Omega^k(M) \to \Omega^{k+1}(M)}_{\text{exterior derivatives}}$ satisfying

ⓐ d is linear over $\mathbb{R}$.

ⓑ $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

ⓒ $d \circ d = 0$

ⓓ For $f \in \Omega^0(M)$, df is the usual differential of f, i.e. $df(V_p) = V_p f$.

Some observations:

① Anything satisfying ⓑ and ⓓ must be local, i.e. $(d\eta)_p$ is determined by $\eta|_U$ for any $p \in U \underset{\text{open}}{\subseteq} M$.

Reason: Take ψ with $\overline{\operatorname{supp} \psi} \subseteq U$ and $\psi = 1$ on $p \in V \overset{\text{open}}{\subseteq} U$

Then
$$d(\psi \wedge \eta) = d\psi \wedge \eta + \psi \cdot d\eta$$
is just $d\eta$ on V and $\operatorname{supp}(\psi\eta) \subseteq U$.

[Thus should be able to compute in local coor.]

② On $\mathbb{R}^n$ these props completely det $d\omega$. For example,

$$d(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = 0 \quad \text{since } d(dx_i) = 0$$

where here $x_i : \mathbb{R}^n \to \mathbb{R}$

and so if $\omega \in \Omega^2(\mathbb{R}^3)$ is

$$\omega = (x+z^2)\, dx \wedge dy + e^y\, dx \wedge dz$$

then

$$d\omega = \underbrace{(d(x+z^2))}_{dx + 2z\,dz} \wedge dx \wedge dy + \underbrace{(de^y)}_{e^y dy} \wedge dx \wedge dz$$

$$= 2z\, dz \wedge dx \wedge dy + e^y\, dy \wedge dx \wedge dz$$

$$= (2z - e^y)\, dx \wedge dy \wedge dz.$$

Two approaches to proving Thm:

Ⓐ Show well-defined for $\Omega^*(\mathbb{R}^n)$

Show that for $F : U \to V$ a diffeomorphism we have $F^*(d\omega) = d(F^*\omega)$. ☆

Define d on M via charts. This fact gives that it is well defined.

See text for details.

B Coordinate-free definition: $\omega \in \Omega^1(M)$

Define $d\omega \in \Omega^2(M)$ by the property that if $X, Y \in \mathfrak{X}(M)$ then

Note clear antisymmetry, bilinearity

$$d\omega(X,Y) = \underbrace{X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])}_{f_{X,Y} \,\in\, C^\infty(M)}$$

<u>Claim</u>: $f_{X,Y}(p)$ only depends on X_p and Y_p.

<u>Idea</u>: Enough to show $f_{\varphi X, Y} = \varphi f_{X,Y}$, which is an easy calculation.

In general, for $\omega \in \Omega^k(M)$ define

means remove

$$d\omega(X_0, X_1, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i\left(\omega(X_1, \dots, \widehat{X_i}, \dots, X_k)\right)$$

$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k)$$

For details on A and B, see Lee.

<u>Prop</u>: $F: M \to N$, $\omega \in \Omega^k(N)$. Then $d(F^*\omega) = F^*(d\omega)$

Lie derivatives of differential forms: $V \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$. Define $\mathcal{L}_V \omega \in \Omega^*(M)$ by

$$(\mathcal{L}_V \omega)_p = \frac{d}{dt}\Big|_{t=0} (\theta_t^* \omega)_p \quad \text{in } \Lambda^k T_pM$$

where θ is the flow assoc to V.

Alternatively, using the same ideas as the proof that $L_X Y = [X,Y]$ it follows that for any $X_1, \dots, X_k \in \mathfrak{X}(M)$

$$\mathcal{L}_V \omega (X_1, \dots, X_k) = V(\omega(X_1, \dots, X_k)) - \omega([V, X_1], X_2, \dots, X_k) - \dots - \omega(X_1, X_2, \dots, [V, X_k])$$

<u>Prop:</u> $V \in \mathfrak{X}(M)$ and $\omega, \eta \in \Omega^*(M)$, then

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)$$

<u>Cartan's Magic Formula</u>: $\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega)$

Here, the interior product $V \lrcorner \omega$ is the $k-1$ form defined by $(V \lrcorner \omega)(\underbrace{W_1, \dots, W_{k-1}}_{\in T_pM}) = \omega(V_p, W_1, \dots, W_{k-1})$