Lecture 21: Prelude to integration

The story so far: M^n smooth, $F: M \to N$ smooth

$T_p M = \begin{Bmatrix} \text{directional} \\ \text{derivatives} \end{Bmatrix} = \begin{Bmatrix} \text{velocity} \\ \text{vectors of} \\ \text{curves} \end{Bmatrix}$, $dF: TM \to TN$,

Vector fields, $[X, Y]$, ... [All about derivatives.]

[On first day, said a smooth mfld something locally like $\mathbb{R}^n$ on which we can do calculus. [Query.]
So far mostly just been differentiation, but there's one exception.]

Next focus: What can we integrate on a smooth manifold? [Need some kind of additional str.]

All diffeomorphic, so "the same" to us.

[Let's think back to vector calculus...]

C curve in $\mathbb{R}^3$. $\quad \gamma : [a,b] \longrightarrow C$

parameterization

Riemannian notion

Both generalize to surfaces, etc.

$$\text{Length}(C) = \int_a^b |\gamma'(t)|\, dt = \int_C ds$$

$f : C \rightarrow \mathbb{R}$ define

$$\int_C f\, ds = \int_a^b f(\gamma(t))\,|\gamma'(t)|\, dt$$

F vector field

Differential forms

Our focus, leads to general Stokes thm

$$\int_C F \cdot ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_C (F \cdot T)\, ds$$

T = unit tangent vector field

Query: What are some differences?

One is that $\int_C F \cdot ds$ depends on the orientation of C

First some algebra...

V vector space [over $\mathbb{R}$]. Consider the dual

$V^* = \{\alpha : V \to \mathbb{R} \mid \alpha \text{ linear}\}$ "space of covectors."

which is also a vector space: $\alpha, \beta \in V^*$ define

$\alpha + \beta \in V^*$ by $(\alpha+\beta)(v) = \alpha(v) + \beta(v)$.

<u>Key facts:</u> Assume V is finite dimensional

① $\dim V^* = \dim V$

If $e_1, \ldots, e_n$ is a basis for V, define $\omega^i \in V^*$

by $\omega^i(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j. \end{cases}$ [Good exercise to see this is a basis.]

② If $F : V \to W$ is a linear transformation,

get $F^* : W^* \to V^*$ by $F^*(\beta) = \beta \circ F$

$W \to \mathbb{R}$ $\quad$ $V \to \mathbb{R}$

If $G : W \to X$, then

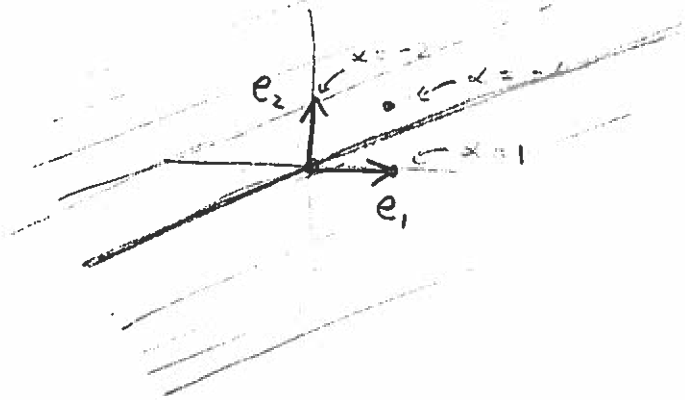
$(G \circ F)^* = F^* \circ G^*$

$$\begin{tikzcd} X^* \arrow[r, "G^*"'] \arrow[rr, bend left, "(G\circ F)^*"] & W^* \arrow[r, "F^*"'] & V^* \end{tikzcd}$$

③ There is a nat'l isomorphism

$V \to (V^*)^*$ given by $v \mapsto \gamma_v$

where $\underset{\curvearrowleft V^*}{\gamma_v(\beta)} = \beta(v)$.



<u>Ex</u>: e_1, e_2 std basis for $\mathbb{R}^2 = V$ $\to$ ω^1, ω^2 dual basis

$\alpha = \omega^1 - 2\omega^2$

$\ker \alpha = \langle 2e_1 + e_2 \rangle$

$f_1 = e_1 \qquad f_2 = e_1 + e_2 \qquad \eta^1, \eta^2$ dual basis

<u>IMPORTANT</u>: $e_i \leftrightarrow \omega^i$ and $f_i \leftrightarrow \eta^i$

induce <u>different</u> maps $V \longleftrightarrow V^*$.

$$\text{equal vectors}\left\{\begin{matrix} e_1 - 2e_2 \\ \| \\ 3f_1 - 2f_2 \end{matrix}\right. \qquad \left.\begin{matrix} \omega^1 - 2\omega^2 \\ \| \\ \eta^1 - \eta^2 \end{matrix}\right\}\text{equal covectors.}$$

$\alpha = a\eta^1 + b\eta^2$

$\alpha(f_1) = a \qquad \alpha(f_1) = \alpha(e_1) = 1$

$\alpha(f_2) = b \qquad \alpha(f_2) = \alpha(e_1 + e_2) = -1$

M smooth n-mfld. The cotangent space to M at p is $T_p^*M = (T_pM)^*$. The cotangent bundle $T^*M = \coprod_p T_p^*M$.

<u>Local coordinates</u>: $p \in \mathbb{R}^n$ coor $(x_1, x_2, \dots, x_n)$

T_pM has basis $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p$

The dual basis for T_p^*M is denoted $dx_1|_p, \dots, dx_n|_p$

<u>Covector fields / 1-forms</u>: $\omega : (p \in M) \longmapsto (\omega_p \in T_p^*M)$

<u>Example</u>: $f : M \to \mathbb{R}$. Consider ω defined by

$\omega_p(V_p) = V_p f$ Normally called "df."

<u>Example</u>: $(x \cos y)\, dx + e^{x+y} dy$