Lecture 23: Integrating covector fields

$$\Omega^1(M) = \left\{ \begin{array}{c} \text{smooth covector fields} \\ (p \in M) \longmapsto (\omega_p \in T_p^* M) \end{array} \right\}$$

$F: M \to N$ gives $F^*: \Omega^1(N) \to \Omega^1(M)$

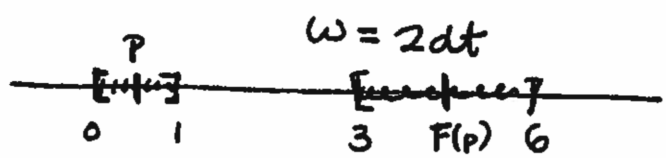
$f \in C^\infty(M)$ has differential $df \in \Omega^1(M)$ defined by $df = f^*(dt)$ or $df(v_p) = v_p f$.

Integration: $[a,b] \subseteq \mathbb{R}$ bounded interval. For $\omega \in \Omega^1([a,b])$ set $\int_{[a,b]} \omega = \int_a^b f(t)\,dt$ where $\omega_t = f(t)\,dt$ for $t \in [a,b]$.

Thm: Suppose $F: [c,d] \to [a,b]$ is a diffeomorphism. Then $\forall \omega \in \Omega^1([a,b])$ we have

$$\int_{[a,b]} \omega = \int_{[c,d]} F^*\omega$$

Motivation:



$F(t) = 3t + 3$

$$(F^*\omega)_p\left(\frac{\partial}{\partial t}\Big|_p\right) = \omega_{F(p)}\left(dF\left(\frac{\partial}{\partial t}\Big|_p\right)\right) = \omega_{F(p)}\left(3\frac{\partial}{\partial t}\Big|_{F(p)}\right) = 6$$

So $F^*\omega = 6\,dt$

$$\int_{[a,b]} \omega = \int_3^6 2\,dt = \boxed{6} \qquad \int_{[c,d]} F^*\omega = \int_0^1 6\,dt = \boxed{6} \checkmark$$

In general, if $\omega = C\,dt$ and F is affine, get

$$F^*\omega = C \cdot F' = C\,\frac{b-a}{d-c} \quad \text{and so}$$

$$\int_{[a,b]} \omega = C(b-a) = \int_{[c,d]} F^*\omega.$$

[Could easily turn this into a proof...]

<u>Proof</u>: $$\int_{[c,d]} F^*\omega = \int_c^d \omega(F(t))\,F'(t)\,dt = \int_a^b \omega(s)\,ds = \int_{[a,b]} \omega. \quad \blacksquare$$

$$(F^*\omega)\left(\frac{\partial}{\partial t}\Big|_t\right) = \omega_{F(t)}\left(dF\left(\frac{\partial}{\partial t}\Big|_t\right)\right)$$

$$dF\left(\frac{\partial}{\partial t}\Big|_t\right) = F'(t)\frac{\partial}{\partial t}\Big|_{F(t)}$$

Suppose $\omega \in \Omega^1(M^n)$ and $F : [a,b] \longrightarrow M$ smooth. Then can consider $\int_{[a,b]} F^*\omega$

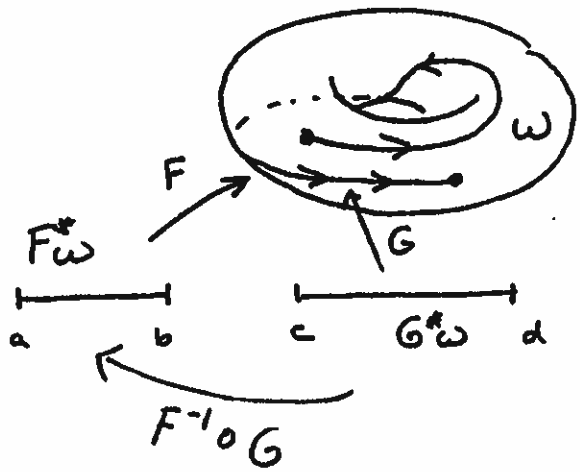
If $C \subseteq M$ is an oriented embedded arc (i.e. $F : [a,b] \hookrightarrow M$ an immersion), then define $\int_C \omega = \int_{[a,b]} F^*\omega$.

Makes sense because

$(F^{-1}\circ G)^*(F^*\omega)$

$= (F\circ(F^{-1}\circ G))^*\omega$

$= G^*\omega.$



[Now need to introduce something from which to build integrable things on higher dimensional manifolds...]

$V_1, \ldots, V_k$ vector spaces. W vector space.

A $F: V_1 \times V_2 \times \cdots \times V_k \longrightarrow W$ is

<u>multilinear</u> if it is linear in each input separately:

$$F(v_1, \ldots, v_{i-1}, av_i + a'v_i', v_{i+1}, \ldots, v_k)$$
$$= a\,F(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k)$$
$$+ a'\,F(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k)$$

[When $k=2$, such a map is called <u>bilinear</u>.]

Examples:

Dot product: $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$
$(v, w) \longmapsto \sum v_i w_i$

Cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Lie bracket:
$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$
$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

Determinant:

$$\det : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{rows of } n \times n \text{ matrix}} \longrightarrow \mathbb{R}$$

Pf: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$ $\det A = \sum (-1)^{i+1} a_{1i} \det \hat{A}_{1i}$

[So works for 1^{st} row and have similar expansions for any other...]

Bilinear forms $V \times V \longrightarrow \mathbb{R}$

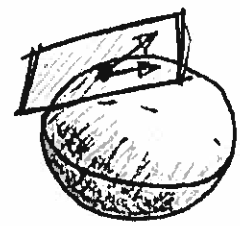
Two nice types: Symmetric: $F(v,w) = F(w,v) \ \forall v, w \in V$
Antisymmetric: $F(v,w) = -F(w,v) \ \forall v, w \in V$

[Lable the examples above.]

$\det : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is antisymmetric.

[Here are some geometric applications of these...]

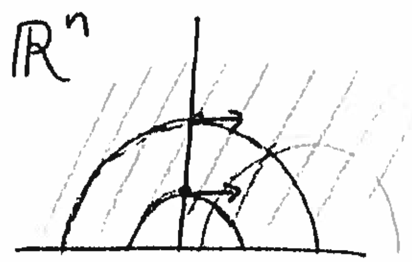
M smooth. A <u>Riemannian metric</u> g on M is a smooth choice of a symmetric positive definite ($g_p(v,v) > 0$ for $v \neq 0$.) bilinear form $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ for each $p \in M$.



<u>Ex</u>: $\mathbb{R}^n$, g_p = dot product

<u>Ex</u>: $S \subseteq \mathbb{R}^n$ smooth embedded submanifold

g_p = restriction of dot product to $T_pS \subseteq T_p\mathbb{R}^n$



<u>Ex</u>: $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$$g_{(x,y)} = \frac{1}{y^2} g_{\mathbb{R}^2}$$

$$g\left(\frac{\partial}{\partial x}\Big|_{(0,1)}, \frac{\partial}{\partial x}\Big|_{(0,1)}\right) = 1$$

$$g\left(\frac{\partial}{\partial x}\Big|_{(0,2)}, \frac{\partial}{\partial x}\Big|_{(0,2)}\right) = \frac{1}{4}$$

If $\gamma : [a,b] \longrightarrow M$ is a curve,

define

$$\text{Length}(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}\, dt$$

and make M into a metric space by

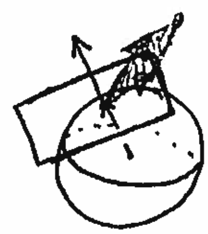
$$d(p,q) = \inf\left\{ \text{Length}(\gamma) \;\middle|\; \gamma \text{ a smooth path from } p \text{ to } q \right\}$$

A 2-form ω on M is a choice of an anti-symmetric bilinear form $\omega_p : T_pM \times T_pM \to \mathbb{R}$ for each $p \in M$. Set $\Omega^2(M) = \left\{ \begin{matrix} \text{all smooth} \\ \text{2-forms} \end{matrix} \right\}$

<u>Ex:</u> $M = \mathbb{R}^2$ $\qquad \omega_p(v, w) = \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

= Signed area of 

<u>Ex:</u> $S^2 \subseteq \mathbb{R}^3 \qquad T_pS^2 \subseteq T_p\mathbb{R}^3$



$$\omega_p(v,w) = \det \begin{pmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

<u>Next:</u> Integrating 2-forms and n-forms.