Lecture 24: Riemannian Geometry

Bilinear Functions: $V \times V \xrightarrow{F} W$ linear in each entry

Symmetric: $F(x,y) = F(y,x) \quad \forall x, y \in V$

Dot product: $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \sum x_i y_i$$

Antisymmetric: $F(x,y) = -F(y,x)$

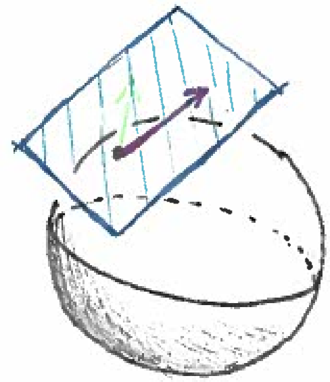
Cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Lie bracket: $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$

A **Riemannian metric** g on M^n is a smooth choice of symmetric positive definite bilinear form $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ for each $p \in M$

Here positive definite means that $g_p(v_p, v_p) \neq 0 \ \forall v_p \neq 0$ in T_pM; by linear algebra, there exists a basis $e_1, \dots, e_n$ for T_pM where $g_p(e_i, e_j) = \delta_{ij}$

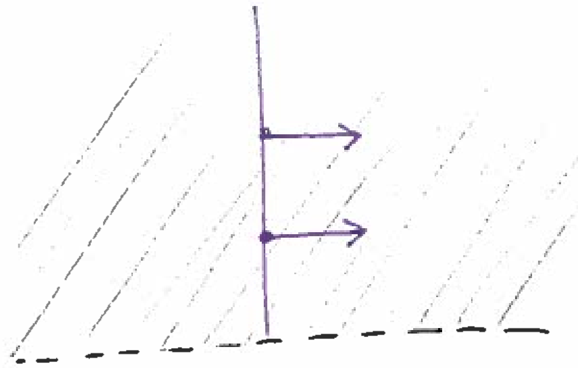
Ex: $\mathbb{R}^n$, g_p = dot product on $T_p\mathbb{R}^n$.



Ex: $S \subseteq \mathbb{R}^n$ embedded submanifold

g_p = restriction of dot product to $T_pS \subseteq T_p\mathbb{R}^n$

Ex: $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$



$$g_{(x,y)} = \frac{1}{y^2}\, g_{dot}$$

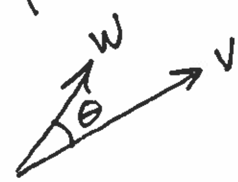
$$g_{(0,1)}\left(\frac{\partial}{\partial x}\Big|_{(0,1)}, \frac{\partial}{\partial x}\Big|_{(0,1)}\right) = 1$$

$$g_{(0,2)}\left(\frac{\partial}{\partial x}\Big|_{(0,2)}, \frac{\partial}{\partial x}\Big|_{(0,2)}\right) = \frac{1}{4}$$

A Riemannian metric lets us talk about

Lengths: $|v_p| = \sqrt{g_p(v_p, v_p)}$ for $v_p \in T_pM$

Angles: $g_p(v_p, w_p) = |v_p|\,|w_p| \cos\theta$ defines where $\theta \in [0, \pi]$

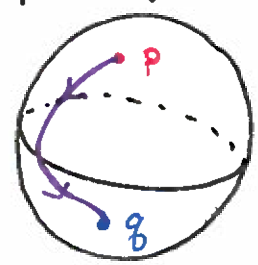


If $\gamma : [a,b] \longrightarrow M$ is a smooth curve,

define
$$\text{Length}(\gamma) = L(\gamma) = \int_a^b |\gamma'(t)| \, dt$$
$$= \int_a^b \sqrt{g_p(\gamma'(t), \gamma'(t))} \, dt$$

and define

$$d(p,q) = \inf\left\{ L(\gamma) \;\middle|\; \begin{array}{l}\gamma \text{ a piecewise smooth} \\ \text{curve from } p \text{ to } q\end{array} \right\}$$



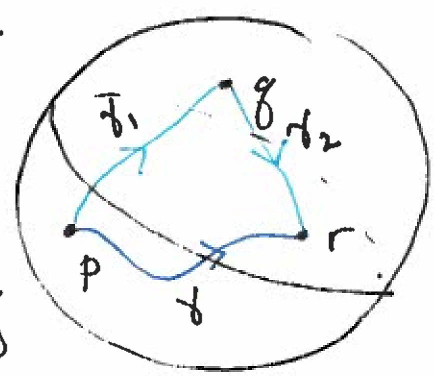
Thm: If g is a Riemannian metric on M, then d is a metric on M.

Pf: [Query: What do we need to check?]

$d(p,p) = 0$: const path from p to p has length 0.

$d(p,q) = d(q,p)$: length of path doesn't depend on its direction

Triangle inequality:



$$d(p,r) = \inf\left\{ L(\gamma) \;\middle|\; \begin{array}{l}\gamma \text{ curve} \\ \text{from } p \text{ to } r\end{array} \right\}$$
$$\leq \inf\left\{ L(\gamma) \;\middle|\; \begin{array}{l}\gamma \text{ curve from } p \text{ to } q \\ \text{to } r.\end{array} \right\}$$

$$= \inf \left\{ L(\gamma_1) + L(\gamma_2) \;\middle|\; \begin{matrix} \gamma_1 \text{ curve from } p \text{ to } q \\ \gamma_2 \text{ curve from } q \text{ to } r \end{matrix} \right\}$$

$$= d(p,q) + d(q,r).$$

<u>$d(p,q) = 0 \Rightarrow p = q$:</u> See text, but idea is that on very small scale any Riemannian metric looks Euclidean.

$\square$

[A lot of fun geometry here...]

<u>Writing down g:</u> [First, some algebra...]

V vector space $\quad V^* \ni \alpha, \beta$

<u>Bilinear form:</u> $\alpha \otimes \beta : V \times V \longrightarrow \mathbb{R}$ $\quad$ [Check this is bilinear!]

$$(x, y) \longmapsto \alpha(x)\beta(y)$$

<u>Consider:</u>

$$BF(V) = \{ \text{set of bilinear forms } V \times V \to \mathbb{R} \}$$

$$= L(V, V; \mathbb{R}) \text{ in Lee's notation}$$

If $e_1, \dots, e_n$ is a basis for V, let $\omega^1, \dots, \omega^n$ be the dual basis for V^*.

Thm: $BF(V)$ has basis $\omega^i \otimes \omega^j$ for $1 \le i,j \le n$.
In particular, $\dim(BF(V)) = (\dim V)^2$

Ex: $n=2$: $2\omega^1\otimes\omega^1 + 3\omega^1\otimes\omega^2 - 5\omega^2\otimes\omega^1 + \omega^2\otimes\omega^2 = g$

$$g(e_1 - e_2, 3e_1) = g(e_1, 3e_1) - g(e_2, 3e_1)$$
$$= 3\,g(e_1,e_1) - 3\,g(e_2,e_1) = 3\cdot 2 - 3\cdot(-5)$$
$$= 21$$

Basic linear alg form:

$$g(x,y) = x^t \begin{pmatrix} 2 & 3 \\ -5 & 1 \end{pmatrix} y$$

column vectors

Pf of thm: Given $g \in BF(V)$ define
$f = \sum_{i,j} g(e_i, e_j)\, \omega^i \otimes \omega^j$. Easy to check that
$f = g$ as functions $V \times V \to \mathbb{R}$. So $\{\omega^i \otimes \omega^j\}$
span. If $\sum_{i,j} a_{ij}\, \omega^i \otimes \omega^j = 0$, then
evaluate on (e_i, e_j) to get $a_{ij} = 0$. So they're also linearly independent.

<u>Example</u>: $(\mathbb{R}^2, g)$

$T_p\mathbb{R}^2$ has basis $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

$T_p^*\mathbb{R}$ has basis dx and dy

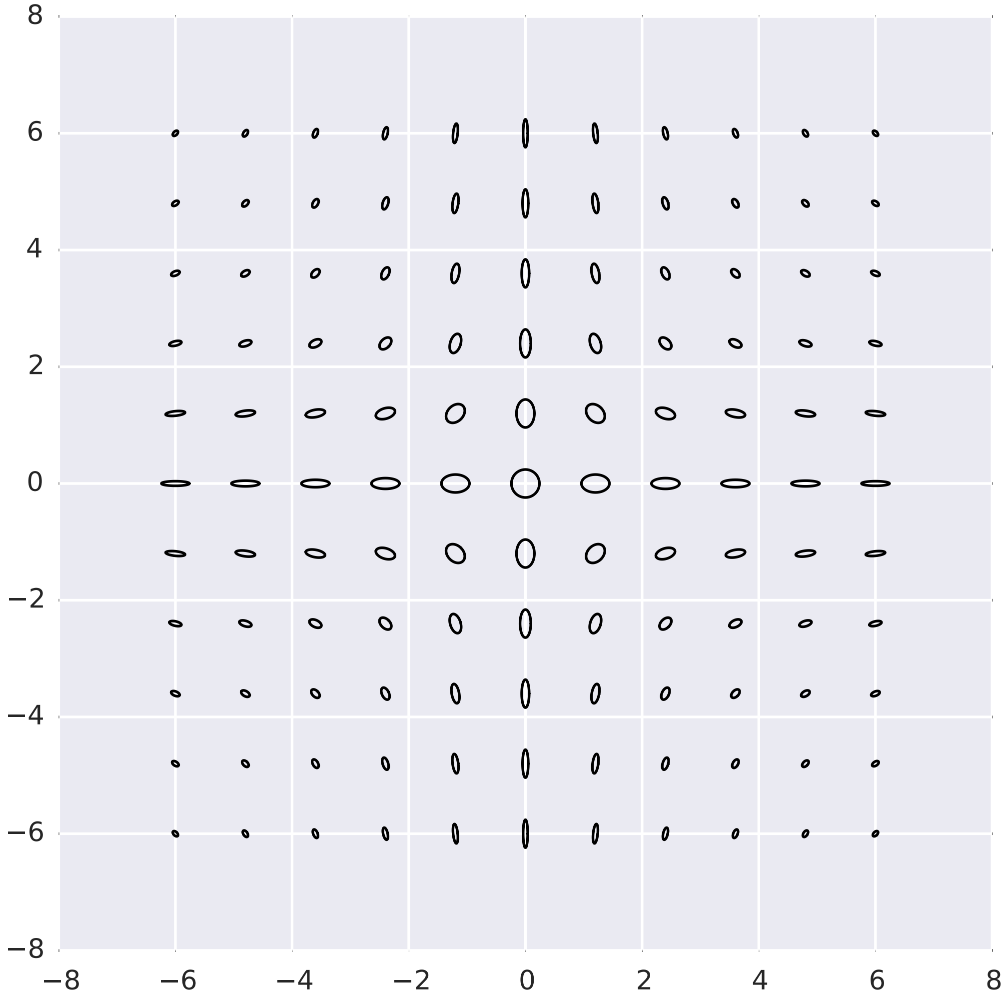
Define

$$g_{(x,y)} = (1+y^2)\, dx\otimes dx + \frac{xy}{2}(dx\otimes dy + dy\otimes dx) + (1+x^2)\, dy\otimes dy$$

which is pos. def. everywhere.

See next page for visulalization

Secretly, this metric comes from the surface $z = xy$ in $\mathbb{R}^3$.



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