# Lecture 25: More Riemannian Geometry



**Riemannian metric:** Smooth choice of $g_p : T_pM \times T_pM \to \mathbb{R}$, a symmetric positive definite bilinear form, for each $p \in M$.

**Lengths:** $\gamma : [a,b] \to M$ curve $\quad L(\gamma) = \int_a^b |\gamma'(t)|\, dt$

$$|\gamma'(t)| = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}$$

**Distances:**

$$d(p,q) = \inf \left\{ L(\gamma) \;\middle|\; \begin{matrix}\gamma \text{ a curve from} \\ p \text{ to } q\end{matrix} \right\}$$

**Tensor product:** $\alpha, \beta \in V^*$ define the bilinear form

$$\alpha \otimes \beta : V \times V \to \mathbb{R} \quad \text{by} \quad \alpha \otimes \beta(x,y) = \alpha(x)\beta(y)$$

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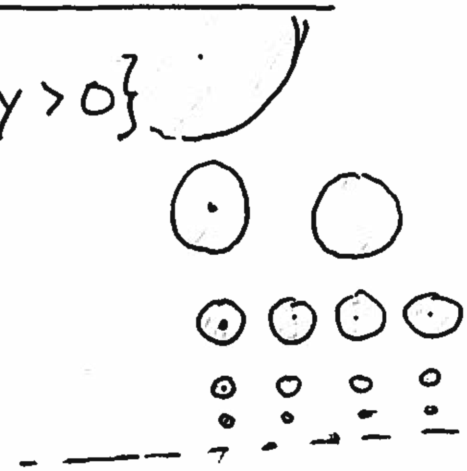
**Examples:** ① $\mathbb{H}^2 = \underset{\text{plane}}{\text{hyperbolic}} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$



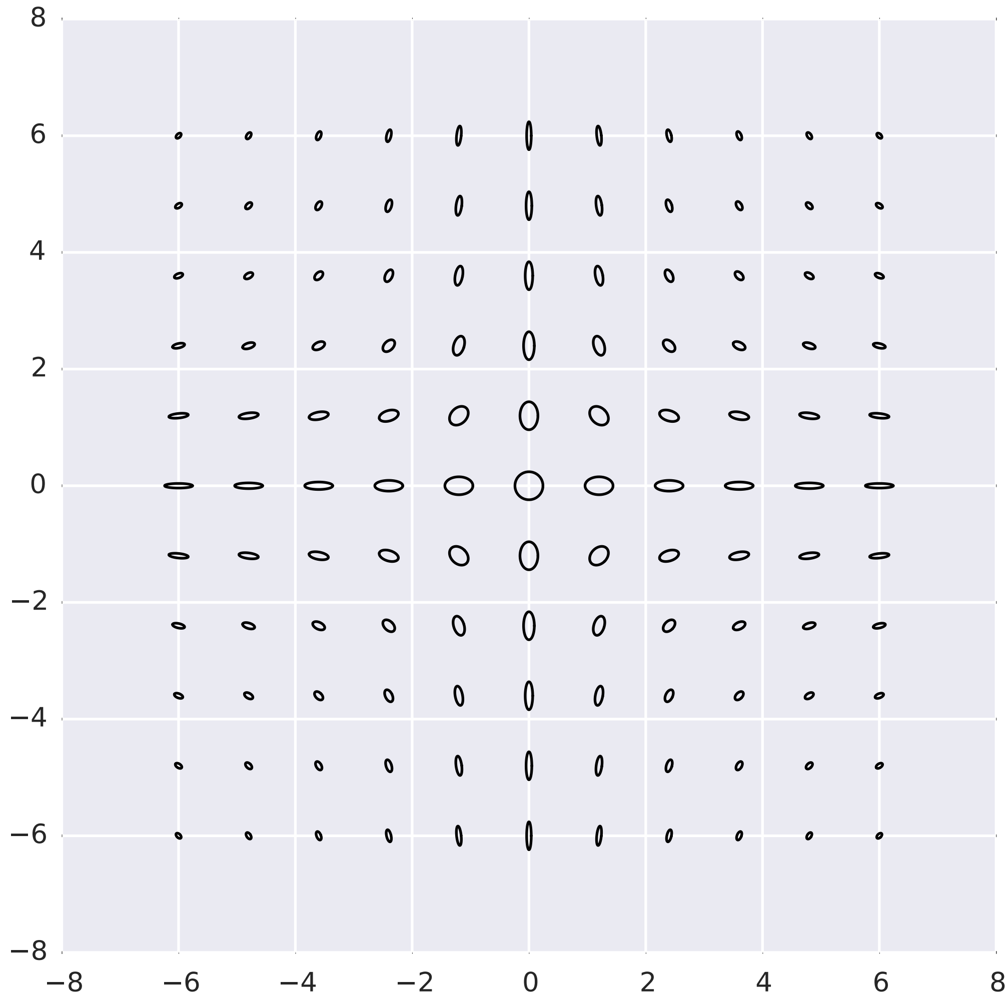
$$g_{(x,y)} = \frac{1}{y^2}\big(\underbrace{dx \otimes dx + dy \otimes dy}_{\substack{\text{Euclidean dot} \\ \text{product}}}\big)$$

$$C_p = \{ v \in T_p\mathbb{H}^2 \mid g_{(x,y)}(v,v) = 1 \}$$

② $$g_{(x,y)} = (1+x^2)\, dx \otimes dx + \frac{xy}{2}\left(dx \otimes dy + dy \otimes dx\right) + (1+y^2)\, dy \otimes dy$$

$M = \mathbb{R}^2$

[Show picture on next page]


8
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8

Remark on smoothness: A vector field $p \mapsto X_p \in T_pM$ is smooth if any of the following equivalent conditions hold:

ⓐ $X : M \to TM$ is smooth, where $TM$ has a particular smooth str.

ⓑ $\forall$ smooth charts $(U, \varphi)$ we have

$$\hat{X} = \sum X_i \frac{\partial}{\partial x_i} \quad \text{where } X_i \in C^\infty(\varphi(U))$$

ⓒ $\forall f \in C^\infty(M)$ the function $Xf : M \to \mathbb{R}$ is smooth. $[$ ⓒ $\Rightarrow$ ⓑ: $X_i = X x_i ]$

Same 3 conditions make sense for Riemannian metrics.

ⓑ $$\hat{g}_{(x_1, \dots, x_n)} = \sum_{i,j} \overbrace{g_{ij}(x_1, \dots, x_n)}^{\text{smooth}} \, dx_i \otimes dx_j$$

ⓒ $\forall X, Y \in \mathfrak{X}(M)$ the fn

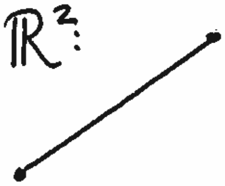
$p \mapsto g_p(X_p, Y_p)$ is smooth.

[ ⓐ Also makes sense, one has a vector bundle over $M$ with fibers $BF(T_pM)$ but will not focus on this for now. ]

[More geometry...] Thm: Every smooth $M^n$ has a Riem. metric.

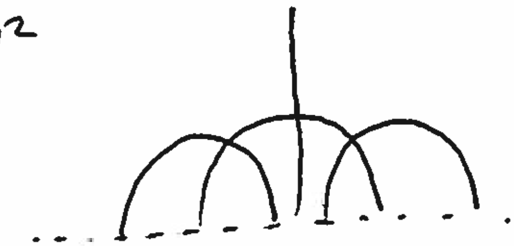
Geodesics: A curve $\gamma$ from $a$ to $b$ is a geodesic if $L(\gamma) = d(a, b)$. [Really should say locally...]

Ex: $\mathbb{R}^2$:



$S^2$:

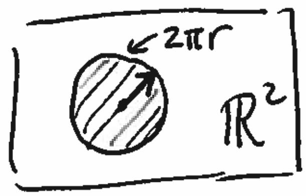
$\mathbb{H}^2$



Volumes: Will discuss soon.

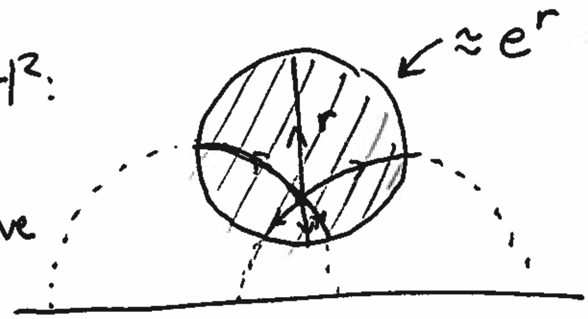
Balls:


positive



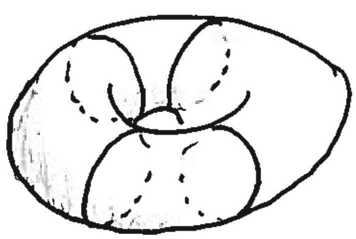

Flat.

$\mathbb{H}^2$:




Negative

Curvature:



$K_p$ is defined by [not that $2^{nd}$ fundamental form!]

$$\text{length}\big(\partial B_r(p)\big) = 2\pi r - \frac{\pi}{3} K(p) r^3 + \text{lower order terms.}$$

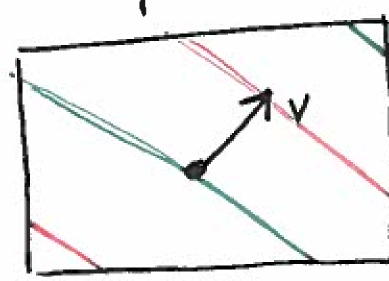
Key feature: A Riemannian metric relates $TM \leftrightarrow T^*M$

Algebra: $V$ vector space. A bilinear form $g : V \times V \to \mathbb{R}$ is non-degenerate if $\forall x \in V\ \exists y, y' \in V$ with $g(x, y) \neq 0$ and $g(y', x) \neq 0$.

Example: $g$ positive definite.

A bilinear form $g$ gives a map $V \to V^*$ by $v \longmapsto \underbrace{(w \mapsto g(v, w))}_{g(v,\cdot)}$

Thm: If $V$ is finite dim'l and $g$ non-degenerate then $v \mapsto g(v, \cdot)$ is an isomorphism.




$g_{dot}(v,\cdot) =$

$g_{dot}(v,\cdot) = 0$

Ex: $T_p\mathbb{R}^n$, $g_{dot}$

$\frac{\partial}{\partial x_i}$ basis.

$g_{dot}\left(\frac{\partial}{\partial x_i}, \cdot\right) \in T_p^*\mathbb{R}^n$ is just $dx_i$.

So, if $g$ is an Riem. metric on $M$, get

$$\begin{aligned} TM &\xrightarrow{\cong} T^*M \\ v_p &\longmapsto g_p(v_p, \cdot) \end{aligned}$$

and hence

$$\begin{aligned} \mathfrak{X}(M) &\longleftrightarrow \Omega^1(M) \\ X &\longmapsto X^\flat \\ \omega^\# &\longmapsfrom \omega \end{aligned}$$

Recall $f : \mathbb{R}^n \to \mathbb{R}$ have $df \in \Omega^1(\mathbb{R}^n)$ given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

which is $\left(\operatorname{grad} f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial}{\partial x_n}\right)^\flat$.

For $f \in C^\infty(M)$ where $M$ is Riemannian, we ~~get~~ define $\operatorname{grad} f \in \mathfrak{X}(M)$ as $(df)^\#$.

[It has all the usual properties....]

**Pseudo-Riemannian:** $g_p$ nondegen sym. bilinear form on $T_pM$.

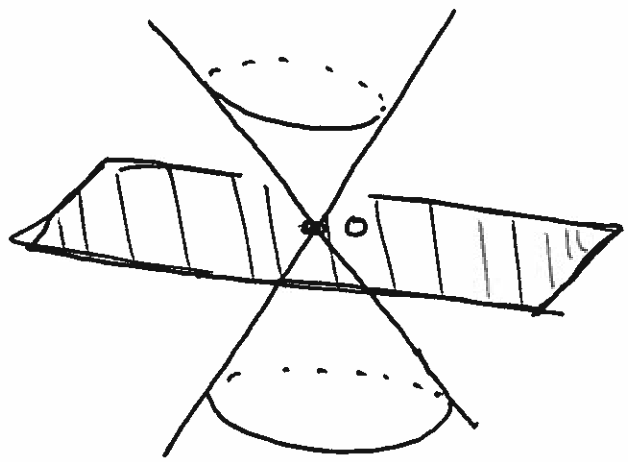
**Lorentzian:** $g((x_1, x_2, x_3), (y_1, y_2, y_3))$

$$= x_1 y_1 + x_2 y_2 - x_3 y_3$$

$g(x,x) = 0$ lightlike

$> 0$ spacelike

$< 0$ timelike




**Lorentzian mfld:** A form of this type at each $p \in M$. The basic object in general relativity.

[If time remains, do hyperboloid model of $\mathbb{H}^2$.]