Lecture 17: Lie bracket and Lie Algebras

M^n smooth, $\mathfrak{X}(M) = \left\{ \begin{array}{c} \text{smooth vector} \\ \text{fields on } M \end{array} \right\} \leftarrow$ Vector space over $\mathbb{R}$

$= [$derivations pg ②$] \nwarrow$ module over $C^\infty(M)$

[Today, give a multiplication, measuring the extent to which the vector fields commute.]

Lie Algebra: A vector space A with a map $A \times A \to A$, denoted $(X, Y) \longmapsto [X, Y]$ where

① Bilinear: $\forall X_1, X_2, Y_1, Y_2 \in A$, $\forall a \in \mathbb{R}$ one has

$$[aX_1 + X_2, Y_1] = a[X_1, Y_1] + [X_2, Y_1]$$
$$[X_1, aY_1 + Y_2] = a[X_1, Y_1] + [X_1, Y_2]$$

② Skew commutive: $[X, Y] = -[Y, X]$

③ $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

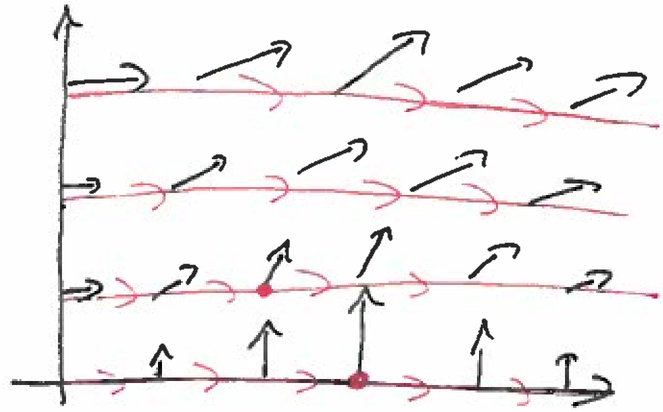
[Jacobi Identity] $\Longleftrightarrow$ $[X, [Y, Z]] - [[X, Y], Z] = [[Z, X], Y]$

Ex: $A = \mathbb{R}^3$ with $[v, w] = v \times w$. [On HW!]

Ex: $A = M_n(\mathbb{R})$ with

$$[X, Y] = XY - YX \quad \text{[Easy check.]}$$

Derivatives of Vector Fields:



$$Y = y\frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y}$$

How does Y change as move along x direction?

$$L_{\frac{\partial}{\partial x}} Y = \cos x \frac{\partial}{\partial y}$$ } Another vector field

What about general M?

Problem: Vector live in different vector spaces!

$X, Y \in \mathfrak{X}(M)$ Let $\theta : (\mathcal{D} \subseteq \mathbb{R} \times M) \longrightarrow M$

be the (partial) flow assoc. to X. That is,

$$\theta_t(m) = \underbrace{\theta^{(m)}}_{\substack{\text{integral curve for } X \\ \text{with } \theta^{(m)}(0) = m}}(t).$$

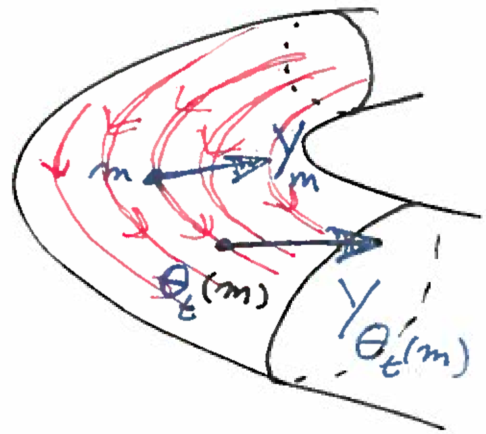
Now define

the Lie derivative $L_X Y \in \mathfrak{X}(M)$ by

$$(L_X Y)_m = \lim_{t \to 0} \frac{1}{t} \left(d\theta_{-t}\left(Y_{\theta_t(m)}\right) - Y_m \right)$$

$$= \frac{d}{dt} \left(d\theta_{-t}\left(Y_{\theta_t(m)}\right) \right) \Big|_{t=0}.$$

[Relate back to example.]



To see this is smooth and that it sat. the Lie algebra axioms, will need to define another way...

$$\mathfrak{X}(M) = \left\{ \begin{array}{l} X : C^\infty(M) \to C^\infty(M) \text{ which is } \mathbb{R}\text{-linear} \\ \text{and sat } X(fg) = (Xf)g + f(Xg) \end{array} \right\}$$

<u>First try</u>: $X, Y \in \mathfrak{X}(M)$ consider $(f \mapsto X(Yf))$

Typically, <u>not</u> a derivation.

<u>Second try</u>: $Z : f \longmapsto X(Yf) - Y(Xf)$

This <u>is</u> a derivation:

$$\begin{aligned} Z(fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X((Yf)g + f(Yg)) - Y((Xf)\cdot g + f\cdot Xg) \\ &= (X(Yf))g + (Yf)(Xg) + (Xf)(Yg) + f(X(Yg)) \\ &\quad - \left(Y(Xf)g + (Xf)(Yg) + (Yf)(Xg) + f(Y(Xg)) \right) \\ &= Z(f)g + fZ(g) \end{aligned}$$

Lie bracket

Define $[X,Y] = Z$. You can easily check that it makes $\mathfrak{X}(M)$ into a Lie algebra.

Thm: $X, Y \in \mathcal{X}(M)$. Then $L_X Y = [X, Y]$

[Will prove later. For now, let's investigate $[X,Y]$...]

Lemma: $[fX, Y] = f[X,Y] - (Yf)X$

Pf: Suppose $g \in C^\infty(M)$. Then

$$[fX, Y]g = (fX)(Yg) - Y(fXg)$$
$$= f\,X(Yg) - (Yf)(Xg) - f\,Y(Xg) \quad \blacksquare$$

$$\Rightarrow [fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

Lemma: For $m \in M$, the vector $[X,Y]_m$ is determined by $X|_U, Y|_U$ for any open $U \ni m$.

If $F: M \to N$ is a diffeo, $X, Y \in \mathcal{X}(M)$, then

$$[F_*(X), F_*(Y)] = F_*([X,Y])$$

Point: Can calculate in local coordinates.

Suppose on U in $\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$ we have

$$X = \sum X_i \frac{\partial}{\partial x_i} \qquad X_i, Y_i : U \to \mathbb{R}$$
$$Y = \sum Y_j \frac{\partial}{\partial x_j}$$

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] f = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} f\right)$$

$$= 0 \text{ since } f \in C^\infty(\mathbb{R}^n)$$

So $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ vector field.

Hence

$f \quad X \quad g \quad Y$ in previous lemma

$$[X, Y] = \sum_{i,j} \left[X_i \frac{\partial}{\partial x_i}, Y_j \frac{\partial}{\partial y_j}\right]$$

$$= \sum_{i,j} X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} X_i \frac{\partial Y_i}{\partial x_j} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$