Lecture 28: Orientations of Manifolds

An orientation of a finite dim'l vector space V is

ⓐ A choice of basis $\mathcal{B}$, where $\mathcal{B} \sim \mathcal{B}'$ if $\det[\mathrm{Id}]_{\mathcal{B}'}^{\mathcal{B}} > 0$

ⓑ A connected component of $\underbrace{\Lambda^{\dim V}(V)}_{\cong \mathbb{R}} \setminus \{0\}$

Note: $\sim$ is an equivalence relation since $[\mathrm{Id}]_{\mathcal{B}''}^{\mathcal{B}} = [\mathrm{Id}]_{\mathcal{B}'}^{\mathcal{B}} [\mathrm{Id}]_{\mathcal{B}''}^{\mathcal{B}'}$.

Correspondence:

$$\mathcal{B} = (b_1, \dots, b_n) \longmapsto \{\omega \in \Lambda^n(V) \mid \omega(b_1, \dots, b_n) > 0\}$$
$$= \{r\, \beta_1 \wedge \dots \wedge \beta_n \mid r > 0\}$$

where β_i is the basis of V^* dual to b_i.

For $\omega \in \Lambda^n(V)$ nonzero:

$$\{r\omega \mid r > 0\} \longmapsto \{\mathcal{B} = \{b_i\} \mid \omega(b_1, \dots, b_n) > 0\}$$

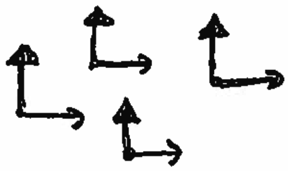
$$r\, \alpha_1 \wedge \dots \wedge \alpha_n \longmapsto (a_1, a_2, \dots, a_n)$$

$$a_i \underset{\text{dual}}{\longleftrightarrow} \alpha_i$$

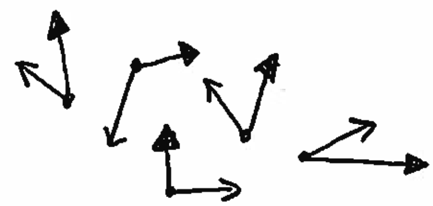
A pointwise orientation on M is choice $\mathcal{O}_p$ of an orientation for each T_pM.

Ex: $M = \mathbb{R}^2$

(a) $\mathcal{O}_p = \left(\frac{\partial}{\partial x}\Big|_p , \frac{\partial}{\partial y}\Big|_p \right)$



(b)



[No assumpt of cont.]

An orientation of M^n is a pointwise orient. where either

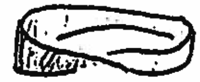
(a) $\forall p \in M$, there is an open nbhd U with vector fields $E_1, \dots, E_n \in \mathfrak{X}(U)$ where $\forall q \in U$ the vectors $E_1|_q, \dots, E_n|_q$ are a basis for T_qM in $\mathcal{O}_q$.

(b) $\exists\, \omega \in \Omega^n(M)$ where $\forall q \in M$ the form ω_q is nonzero and in $\mathcal{O}_q$. [Will prove (a) $\Leftrightarrow$ (b) shortly.]

Ex: $\mathbb{R}^n$ has a standard orientation given by

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \text{ or equivalently } dx_1 \wedge \dots \wedge dx_n$$

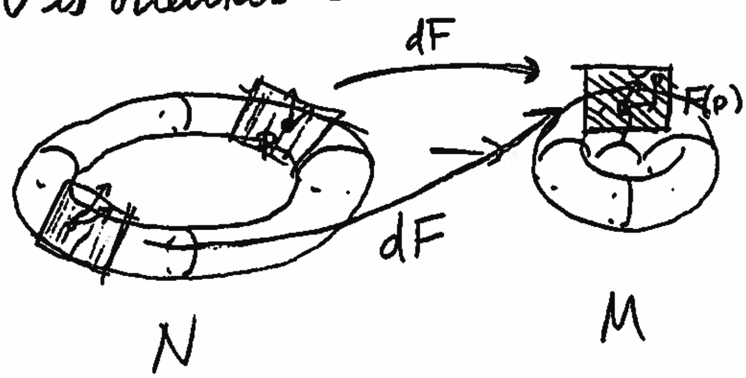
A manifold which admits an orientation, is called orientable: $\mathbb{R}^n$, $\underbrace{S^n, \text{ Lie gp}}_{\text{HW!}}$

Non-orientable: • $\mathbb{RP}^n$ for n even
• Klein bottle • Möbius band 

Fact: Any non-orientable manifold has a 2-fold cover which is orientable, e.g. $S^2 \to \mathbb{RP}^2$

Provisionally, take ⓑ as our definition; the form there is called an orientation form. [MOVE PROP TO HERE.]

Suppose $F: M^n \to N^n$ is a local diffeom. If N is orientable so is M since if $\omega \in \Omega^n(N)$ is an orientation form then $F^*\omega$ is one for M.



[Where am I using that F is a local diffeo?]

If M and N oriented (i.e. we have choosen an orientation) then a local diffeo $F: M \to N$ is orientation preserving: if $F^*\left(\begin{smallmatrix}\text{orientation} \\ \text{form for } N\end{smallmatrix}\right)$ gives the prefered orient at each pt of M.

Ex: $M = N = (\mathbb{R}^n, \text{std orient})$

$\mathrm{id}_{\mathbb{R}^n}$ is orient pres. $\qquad (x_1, x_2, \ldots, x_n) \longmapsto (-x_1, x_2, \ldots, x_n)$

orient. reversing.

Prop: If M is orientable and connected, then M has exactly two orientations.

Pf. If α is some orient form, then $-\alpha$ induces the opposite orient on each T_pM; so there are at least two orient. If β is another orient form, then set $A = \{ p \in M \mid \alpha_p \text{ and } \beta_p \text{ give same orient to } T_pM \}$

Claim: A is open

Given $p \in A$, choose a smooth chart (U, φ) about p with U connected. Then in local coor:

$$\hat{\alpha}_{(x_1, \ldots, x_n)} = \varphi^* \alpha = a(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

$$\hat{\beta}_{(x_1, \ldots, x_n)} = \varphi^* \beta = b(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

where $a, b \in C^\infty(\varphi(U))$ are nowhere vanishing.

Since a, b have the same sign at $\varphi(p)$ they have the same sign on all of $\varphi(U)$. In particular, $U \subseteq A$. The same argument shows that $M \setminus A = \{p \in M \mid -\alpha_p \text{ and } \beta_p \text{ give the same orient to } T_pM\}$ is open. Hence β gives the same orient as either α or $-\alpha$. ▨

<u>Pf of equivalence of ⓐ and ⓑ</u>:

ⓑ $\Rightarrow$ ⓐ: For any orient form α and $p \in M$ choose smooth chart as above where $a > 0$ on $\varphi(U)$. Then $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in \mathfrak{X}(U)$ are the required vector fields.

ⓐ $\Rightarrow$ ⓑ. On next weeks HW, using material from Monday.

If U, V are open sets in $\mathbb{R}^n$ and $F: U \to V$ is a diffeom, then F preserves the std orient if and only if $\det(D_p F) > 0$ for any (every) $p \in U$.

<u>Thm</u>: If $\omega \in \Omega^n(V)$ and F is orient pres, then $\int_V \omega = \int_U F^*\omega$. If F is orient rev, then $\int_V \omega = -\int_U F^*\omega$. [I'm being a little sloppy here as to whether these exist.]

<u>Pf</u>: Change of variables.