Lecture 18: Equality of Lie derivative and Lie bracket

$X, Y \in \mathfrak{X}(M)$. Define $[X,Y] \in \mathfrak{X}(M)$ by

$$[X,Y]f = X(Yf) - Y(Xf)$$

If θ is the flow assoc. to X define for $m \in M$

$$(L_X Y)_m = \lim_{t \to 0} \frac{1}{t}\left(d\theta_{-t}(Y_{\theta_t(m)}) - Y_m \right)$$

$$= \frac{d}{dt} d\theta_{-t}(Y_{\theta_t(m)}) \Big|_{t=0}$$

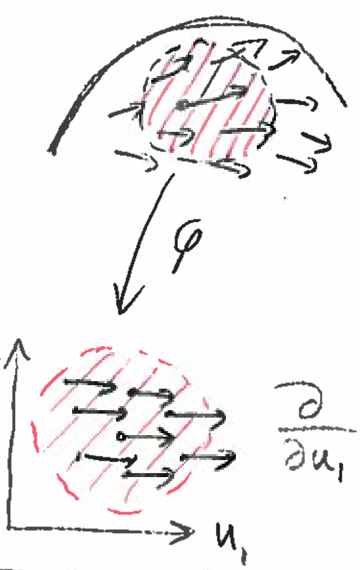
Thm: $[X,Y] = L_X Y$

In coor $(u_1, \dots, u_n)$ if $X = \sum X_i \frac{\partial}{\partial u_i}$ and $Y = \sum Y_i \frac{\partial}{\partial u_i}$ then ☆

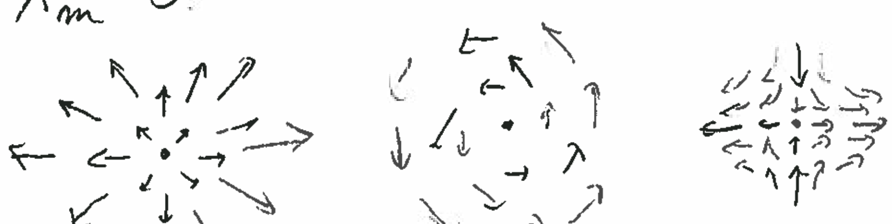
$$[X,Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X_i \frac{\partial Y_j}{\partial u_i} - Y_i \frac{\partial X_j}{\partial u_i} \right) \frac{\partial}{\partial u_j}$$

Lemma: $X \in \mathfrak{X}(M)$

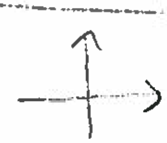
If $X_m \neq 0$, then $\exists$ a chart (U, φ) about m with $\varphi_* X = \frac{\partial}{\partial u_1}$



Note: No standard picture if $X_m = 0$.



$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $\quad -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ $\quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$



$X_m = 0$ for all m.

Pf of Thm assuming lemma: Set

$$\mathcal{R} = \{ m \in M \mid X_m \neq 0 \}$$

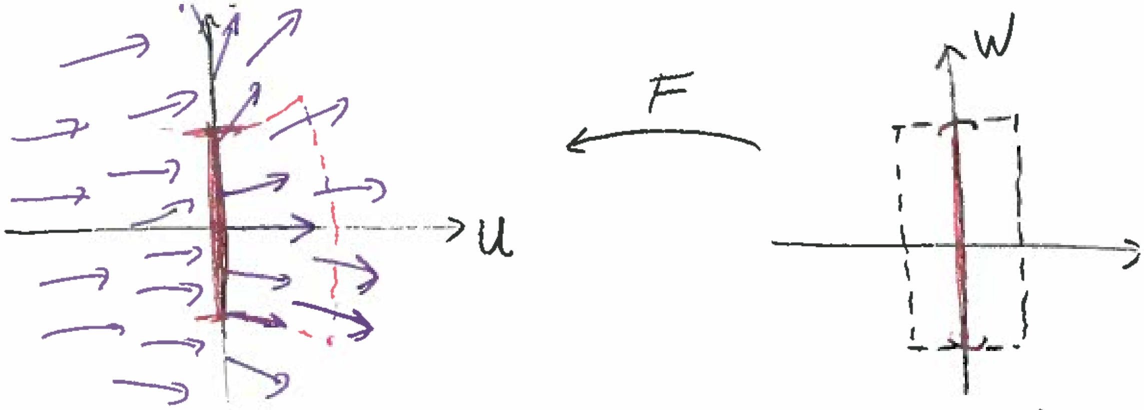
Case $m \in \mathcal{R}$: Choose coor where $X = \frac{\partial}{\partial u_1}$. Then by ⊛ have $[X, Y] = \sum_{j=1}^{n} \frac{\partial Y_j}{\partial u_1} \frac{\partial}{\partial u_j}$. That is, if view $Y : \mathbb{R}^n \to \mathbb{R}^n$ then $[X, Y]$ is just $\frac{\partial Y}{\partial u_1}$.

In these coor, $\theta_t(u_1, \dots, u_n) = (u_1 + t, u_2, \dots, u_n)$ and $d\theta_t =$ "the identity" so $L_X Y$ is also just $\frac{\partial Y}{\partial u_1}$.

Case $m \in \overline{\mathcal{R}}$: True by continuity of $[X, Y]$ and $L_X Y$

Case $m \notin \overline{\mathcal{R}}$: $\exists\, \mathcal{U} \ni m$ where $X|_{\mathcal{U}} = 0$. Then $[X, Y] = 0$ on $\mathcal{U}$ from ⊛. Also $\theta_t|_{\mathcal{U}} = id_{\mathcal{U}}$ for all t, so $\theta_t(m) = m$ and $d(\theta_{-t})_m = id_{T_m M}$. So $(L_X Y)_m = 0$. ▨

Pf of Lemma: Can assume $M = \mathbb{R}^n$ and $m = 0$. For simplicity take $n = 2$. By linear change of coor, can assume $X_0 = \frac{\partial}{\partial u}\Big|_0$.



Choose an open interval $V \ni 0$ in the v-axis so that $X|_V$ is never tangent to V. Choose $\varepsilon > 0$ so that integral curves starting at $p \in V$ exist for $t \in (-\varepsilon, \varepsilon)$

Now define $F : (-\varepsilon, \varepsilon) \times V \longrightarrow \mathbb{R}^2$. Note that
$$(t, w) \longmapsto \theta_t(w)$$

$F(0,w) = (0,w)$ and so $dF\left(\frac{\partial}{\partial w}\Big|_0\right) = \frac{\partial}{\partial v}$ and that $\forall w \in V$ the map $t \mapsto F(t,w)$ is the integral curve for X starting at $(0,w)$; hence $dF\left(\frac{\partial}{\partial t}\Big|_0\right) = \frac{\partial}{\partial u}$. So F is a diff near 0 and $(F^{-1})_*(X) = \frac{\partial}{\partial t}$ as desired. ▨

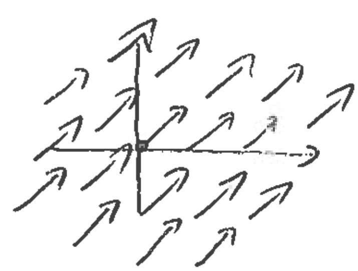
<u>So far:</u> $\mathfrak{X}(M)$ with $[X,Y] = L_X Y$ is a Lie algebra. But $\mathfrak{X}(M)$ is quite big [HW] but for Lie groups there is a finite dim'l subspace which is closed under $[\ ,\ ]$

<u>Thm:</u> G a Lie gp. If $X, Y \in \mathfrak{X}(G)$ are left-invariant, so is $[X,Y]$.

<u>Pf.</u> Fix $g \in G$. Then $(L_g)_*([X,Y]) = [(L_g)_*X, (L_g)_*Y]$
$= [X,Y]$. So $[X,Y]$ is left-invariant.

The Lie algebra $\mathfrak{g}$ of G is the vector sp of left-invariant vector fields. Know that
$$\mathfrak{g} \longrightarrow T_eG$$
$$X \longmapsto X_e$$
is an isomorphism.

<u>Ex:</u> $G = \mathbb{R}^n$ $\quad \mathfrak{g} = \mathbb{R}^n$ with $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$



<u>Ex:</u> $G = GL_n\mathbb{R} \subseteq M_n(\mathbb{R})$

$$\mathfrak{g} = G_e = M_n(\mathbb{R})$$

<u>Fact:</u> Under , one has $[X,Y] = X \cdot Y - Y \cdot X$

[Dreary calculation in textbook, will come back to.]

[Lie algebra is very closely connected to the Lie group. Indeed up to covers, the former determines the latter...]

<u>Prop</u>: $F: G \to H$ a Lie gp homomorphism. Then $dF_e: T_eG \to T_eH$ induces a homomorphism of Lie algebras under the ident $\mathfrak{g} = T_eG$ and $\mathfrak{h} = T_eH$.

<u>Pf</u>: Let $X \in \mathfrak{g}$ and $\bar{X} \in \mathfrak{h}$ have $\bar{X}_e = dF(X_e)$.

<u>Claim</u>: $\forall g \in G$ one has $dF(X_g) = \bar{X}_{F(g)}$.

<u>Reason</u>: $X_g = dL_g(X_e)$, $\bar{X}_{F(g)} = dL_{F(g)}(dF(X_e))$, and $F \circ L_g = L_{F(g)} \circ F$.

To show $dF([X,Y]) = [\bar{X}, \bar{Y}]$, by left-invariance we just need to check this at e_H. Since F has constant rank, this is a straight-forward calculation in local coordinates using $\star$. $\blacksquare$