Lecture 19: More on Lie Algebras of Lie Groups

Remind about Midterm! ☆

Previously: Thm $X, Y \in \mathfrak{X}(M)$. Then $[X,Y] = L_X Y$.

Def. G a Lie gp. The Lie algebra $\mathfrak{g}$ of G is the vector space of all left-invariant vector fields, which is closed under $[\ ,\ ]$. Note $\mathfrak{g} \cong T_e G$.

Ex: [HW!] If G is abelian, then $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$

Ex: $G = GL_n \mathbb{R}$, $\mathfrak{g} = T_e G = M_n(\mathbb{R}) \ni X, Y$, and $[X,Y] = X \cdot Y - Y \cdot X$. [Calculation in book.]

The str of $[\ ,\ ]$ on $\mathfrak{g}$ very closely related to mult on G. Distinct Lie gps can have $\cong$ Lie algebras (e.g. $\mathbb{R}^2$, $S^1 \times S^1$, $S^1 \times \mathbb{R}$), but such groups are "closely related". [Unique simply connected one, covers all the rest.]

Thm: $F: G \to H$ a Lie group homomorphism. Then $dF_e : \underset{\cong \mathfrak{g}}{T_e G} \to \underset{\cong \mathfrak{h}}{T_e H}$ is a homomorphism of Lie algebras.

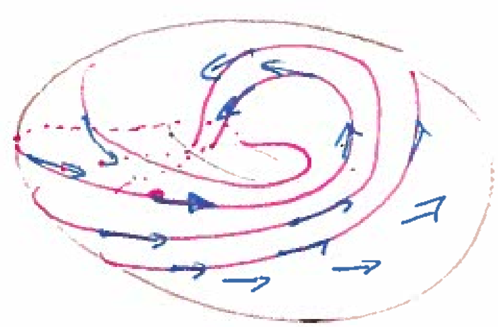
When F is injective (i.e. gives a Lie subgp)
thus have $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}) \cong (dF_e(\mathfrak{g}), [\ ,\ ]_{\mathfrak{h}})$
via dF_e. [Query: Why is ↗ closed under ↑ ?]

Thus a Lie subgp K of H gives a sub-Lie algebra
$\mathfrak{k} = T_eK \subseteq \mathfrak{h}$.

<u>Fact:</u> Any sub Lie algebra of $\mathfrak{h}$ comes from a Lie subgp of H.

<u>Pf of Thm:</u> Given $X \in \mathfrak{g}$ define $\bar{X} \in \mathfrak{h}$ by
$\bar{X}_{e_H} = dF(X_{e_G})$.

<u>Claim:</u> $dF(X_g) = \bar{X}_{F(g)}$ $\forall g \in G$.

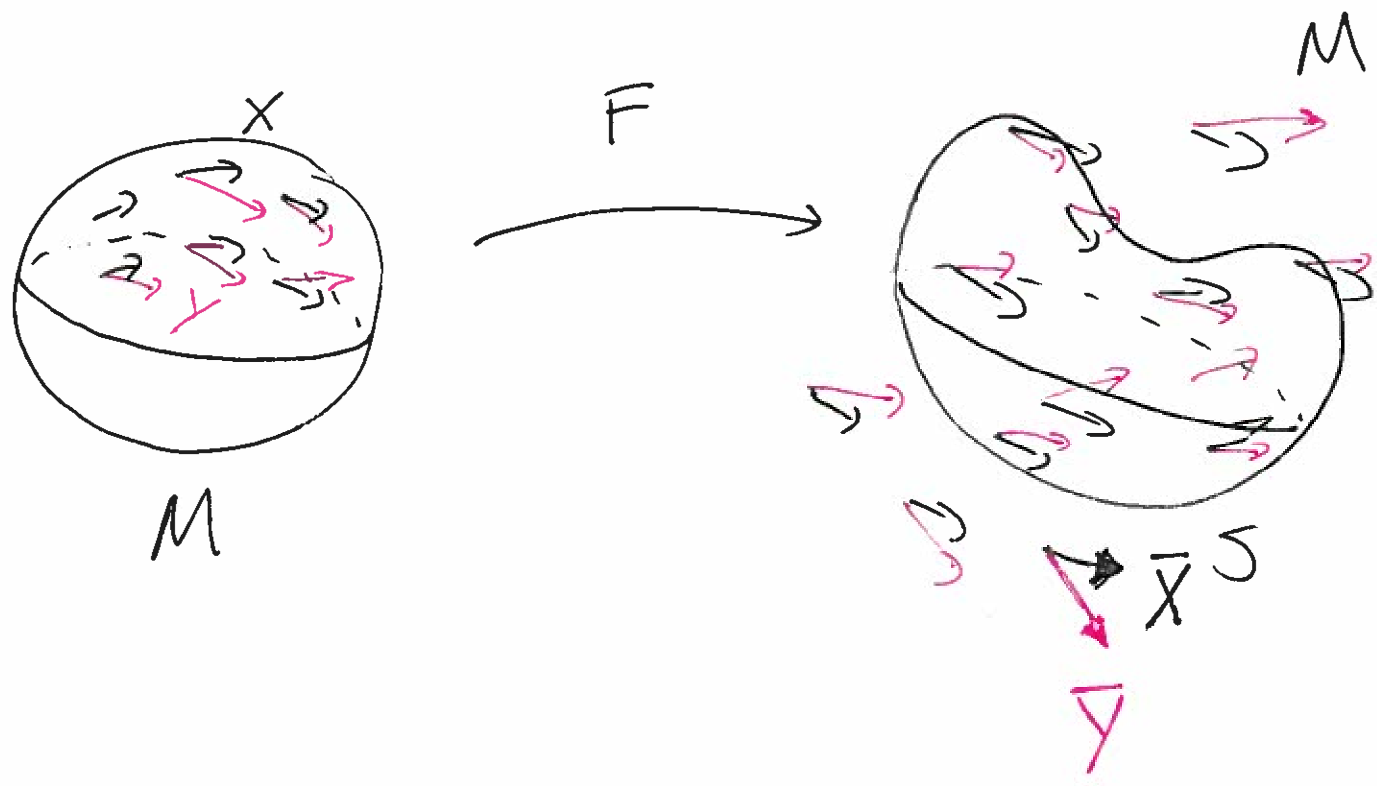


<u>Reason:</u> $X_g = (dL_g)(X_e)$,
$\bar{X}_{F(g)} = dL_{F(g)}(dF(X_e))$, and $F \circ L_g = L_{F(g)} \circ F$.

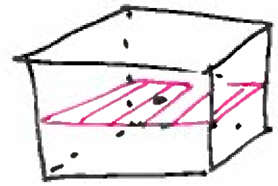
[The book calls such vector fields "F-related".]

For dF_e to be a Lie algebra homom means $\overline{[X,Y]} = [\bar{X}, \bar{Y}]$. This follows from:

<u>Lemma</u>: Suppose $F: M \to N$ is an injective immersion and $\bar{X}, \bar{Y} \in \mathfrak{X}(N)$ are tangent to $S = F(M)$; that is $\forall p \in S$ we have $\bar{X}_p, \bar{Y}_p \in T_p S = dF(T_{F^{-1}(p)} M)$. Then $[\bar{X}, \bar{Y}]$ is also tangent to S and moreover $[\bar{X}, \bar{Y}]|_S$ can be computed in M; that is if $\bar{X}|_S = F_* X$ and $\bar{Y}|_S = F_* Y$ for $X, Y \in \mathfrak{X}(M)$ then $[\bar{X}, \bar{Y}]|_S = F_*([X,Y])$.



Pf of Lemma: ① Use local coor where S becomes a coordinate subspace, apply formula for $[\ ,\ ]$.



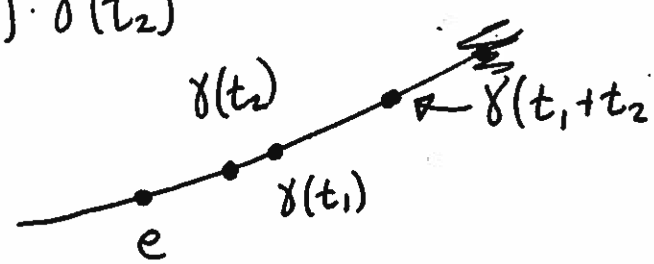
② For simplicity, assume that both X and $\bar{X}$ induce global flows θ and $\bar{\theta}$ on M and N respectively. Surpressing F by regarding it as the identity gives $\bar{\theta}_t(S) = S$ and $\bar{\theta}_t|_S = \theta_t$ for all t. Lemma now clear from regarding $[\bar{X}, \bar{Y}]$ as $L_{\bar{X}} \bar{Y}$.

Let G be a Lie gp. If $L \subseteq \mathfrak{g}$ is a 1-dim'l vector subspace, then L is a sub Lie alg of $\mathfrak{g}$, [Query: Why?] since $[sX, tX] = s \cdot t\,[X,X] = 0$. So there should be a 1-dim'l Lie subgp of G corresponding to L.

Suppose $X \neq 0$ in $\mathfrak{g}$. Let $\gamma : \mathbb{R} \to G$ be the integral curve of X with $\gamma(0) = e$.

__Claim:__ γ is a Lie gp homomorphism.

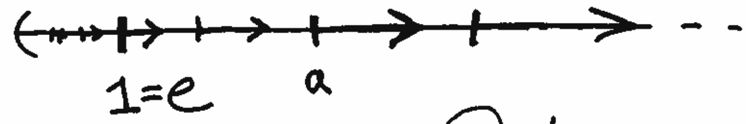
__Pf:__ Let $t_1, t_2 \in \mathbb{R}$. Then $\gamma(t_1)\cdot\gamma(t_2)$ $= L_{\gamma(t_1)}(\gamma(t_2))$. Now



$L_{\gamma(t_1)} \circ \gamma$ is the integral curve for $(L_{\gamma(t_1)})_* X = X$ starting at $\gamma(t_1)$. Hence $(L_{\gamma(t_1)} \circ \gamma)(t) = \gamma(t_1 + t)$ for all t.

In particular, $\gamma(t_1 + t_2) = \gamma(t_1)\cdot\gamma(t_2)$.

<u>Ex</u>: $G = (\mathbb{R}_+, \times)$



Left-invariant vector field assoc. to $\frac{\partial}{\partial x}\big|_1$ is

$X_a = a\frac{\partial}{\partial x}\big|_a$ since $L_a: x \longmapsto ax$.

Integral curve is $t \longmapsto e^t$, a gp homom.

<u>Ex</u>: $G = GL_n\mathbb{R}$ $\quad X \in \mathfrak{g} = T_eG = M_n(\mathbb{R})$.

Define $\exp: \mathfrak{g} \longrightarrow G$ by the following rule

$$\exp(X) = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}X^i$$

[Not clear this makes sense, but in fact it's smooth!]

$\gamma(t) = \exp(tX)$ is the integral curve for X!

<u>Ex</u>: $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad X^n = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & n \text{ odd} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \text{ even} \end{cases}$

$$\exp(tX) = 1 + \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} t^3 & 0 \\ 0 & -t^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + t + \frac{1}{2}t^2 + \cdots & 0 \\ 0 & 1 - t + \frac{1}{2}(-t)^2 + \frac{1}{6}(-t)^3 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$