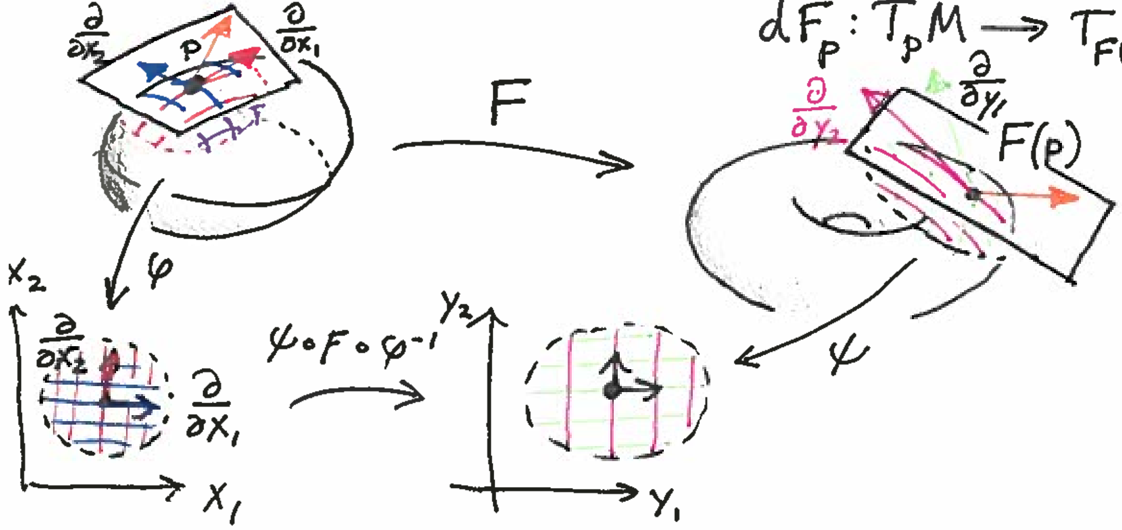
Lecture 7: Immersions, embeddings, and covering maps.

Previously on Math 518... $F: M \longrightarrow N$ smooth

$dF_p : T_pM \longrightarrow T_{F(p)}N$ linear trans.



Immersion: Smooth $F: M \longrightarrow N$ where $\forall p \in M$ the derivative $dF_p : T_pM \longrightarrow T_{F(p)}N$ is injective.

[Query: what are some equiv for this?]

Ex: $F: \mathbb{R} \longrightarrow \mathbb{R}^3$ where $t \longmapsto (\cos t, \sin t, t)$

since the matrix of dF_{t_0} w.r.t $\left\{ \frac{\partial}{\partial t}\Big|_{t_0} \right\}$ and $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}\Big|_{F(t_0)}$ is

just $F'(t_0) = (-\sin t_0, \cos t_0, 1)$

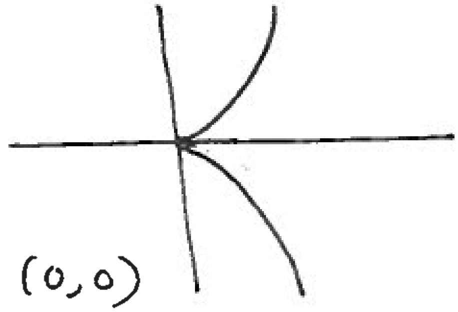
Non Ex: ① $\mathbb{R}^3 \longmapsto \mathbb{R}$, $(x,y,z) \longmapsto z$ $\quad dF_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

More generally, any F where $\dim M > \dim N$.

(2) $\mathbb{R} \to \mathbb{R}^2$

$t \mapsto (t^2, t^3)$

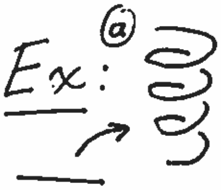
$dF_0 = (2t, 3t^2)\big|_{t=0} = (0,0)$



Topological Embedding: A continuous $f: X \to Y$ (top. spaces) which is a homeomorphism from X to $f(X)$, where the latter has the subspace topology coming from Y.
Equivalently, f is a 1-1 cont fn so that $f^{-1}: f(X) \to X$ is also continuous. [Mark two examples above.]

Smooth Embedding: An immersion which is also a topological embedding.

Ex: (a)

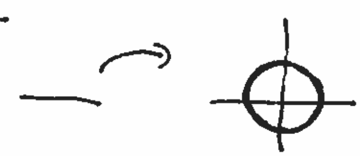


(b) $S^n \to \mathbb{R}^{n+1}$ [Check!]

(c) For open $U \subseteq M$, $i: U \xrightarrow[\text{inclusion}]{} M$

Immersions which are not smooth embeddings:

(a) $\mathbb{R} \to \mathbb{R}^2$ where $t \mapsto (\cos t, \sin t)$

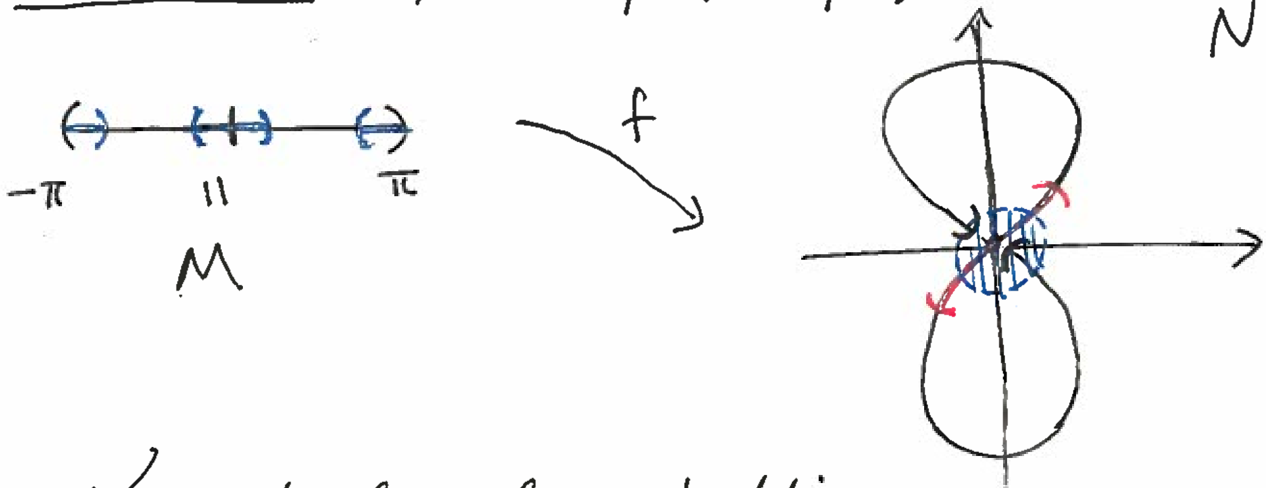


(b) $f: (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin 2t, \sin t)$

This is 1-1. $\qquad f'(t) = (2\cos 2t, \cos t)$

Lemniscate: $x^2 = 4y^2(1-y^2)$

$N = \mathbb{R}^2$



Not a topological embedding, since

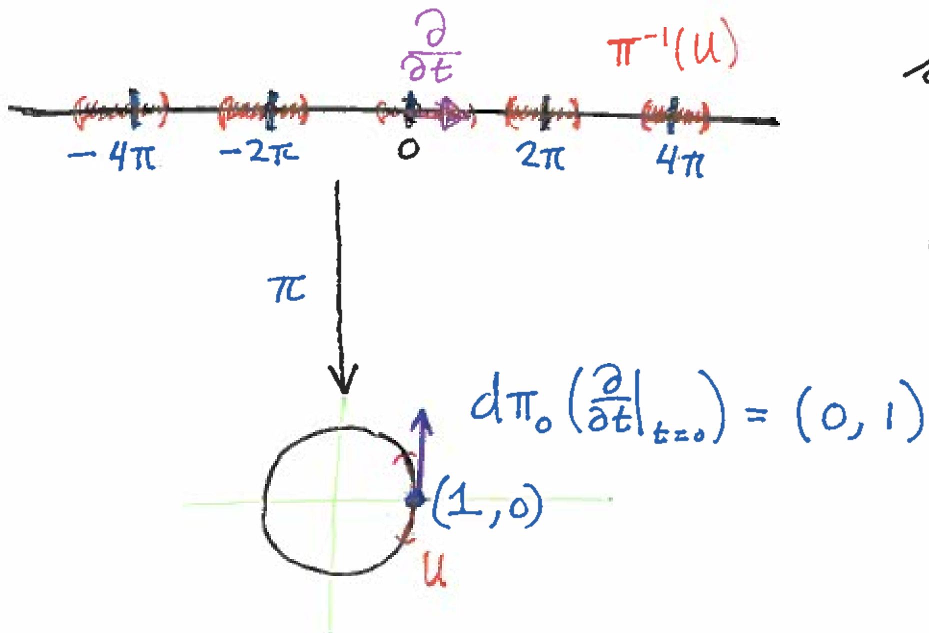
ⓐ $f((-1,1))$ is not open in $f(M)$ since $\nexists$ an open set U in N with $U \cap f(M) = f((-1,1))$. Any open set U containing $(0,0)$ must include nbhds of $\pm\pi$ when pulled back to M.

ⓑ M is not compact but $f(M)$ is.

Def: An <u>embedded/regular submanifold</u> of a smooth manifold M is the image in M of a smooth embedding $S \longrightarrow M$.

Ex: ① $S^n \subseteq \mathbb{R}^n$ ② $M \times \{pt\} \subseteq M \times N$

Consider $\pi : \mathbb{R} \to S^1$, $t \mapsto e^{it}$, a smooth immersion



Note $\pi^{-1}(U) =$ disjoint union of copies of U.

<u>Def</u>: A <u>smooth covering map</u> $\pi : E \to B$ is an onto smooth immersion so that every $b \in B$ has an open nbhd U where $\forall$ connected components $\tilde{U}$ of $\pi^{-1}(U)$ the restriction $\pi|_{\tilde{U}} \to U$ is a diffeomorphism. [In example, smoothness of inverse is consequence of HW prob on angle fns.]

<u>Ex</u>: $\mathbb{R} \xrightarrow[\pi]{} S^1$ with $t \mapsto e^{it}$

<u>Non-Ex</u>: $\pi|_{(-2\pi, 2\pi)}$