Lecture 10: Immersions and submersions in local coordinates; preimages of regular values.

Inverse Function Thm.: Suppose $F : (U \subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is smooth.

If $D_p F$ is invertible for some $p \in U$, then $\exists B = B_r(p) \subseteq U$ so that ① $F|_B$ is 1-1 ② $F(B)$ is open ③ $(F|_B)^{-1}$ is smooth.

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Cor. Suppose $F : M \longrightarrow N$ is a smooth homeomorphism. If dF_p is invertible $\forall p \in M$ then F is a diffeomorphism.

Thm: Suppose $F : M^m \longrightarrow N^n$ is smooth, $p \in M$.

ⓐ If F is an immersion, $\exists$ coordinates (U, φ) of M and (V, ψ) of N so that $\varphi(p) = 0$, $\psi(p) = 0$ and

$$\hat{F}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m})$$

ⓑ If F is a submersion, can arrange that

$$\hat{F}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n)$$

[Lee calls this the "Rank Thm;" includes the broader class of maps of "constant rank."]

<u>Proof</u> Since statement is local, can take $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $p = 0$, $F(p) = 0$.

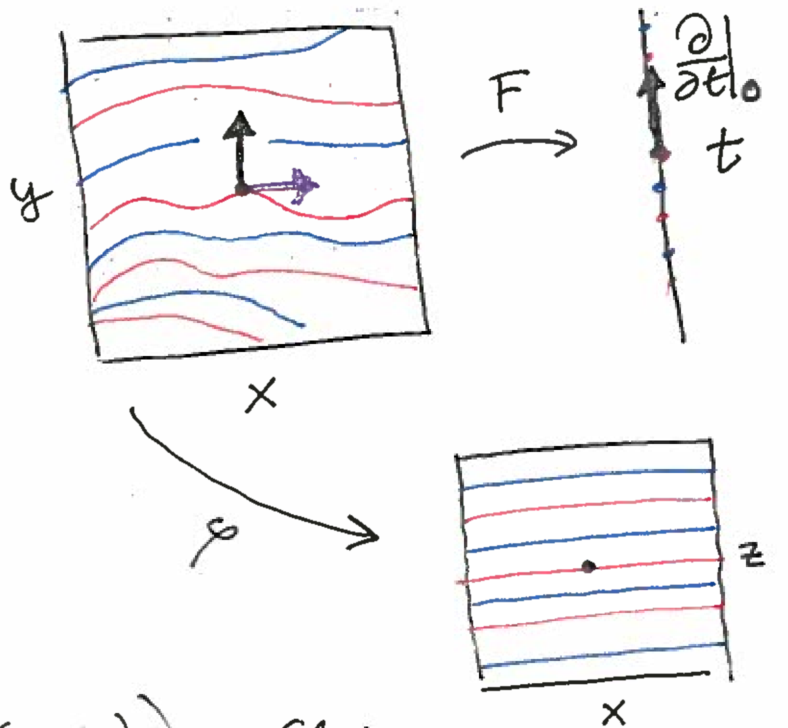
(a) $m = 2$, $n = 1$

Do a linear change of coordinates so that

$$dF_0\left(\frac{\partial}{\partial y}\Big|_0\right) = \frac{\partial}{\partial t}\Big|_0$$

and

$$dF_0\left(\frac{\partial}{\partial x}\Big|_0\right) = 0.$$



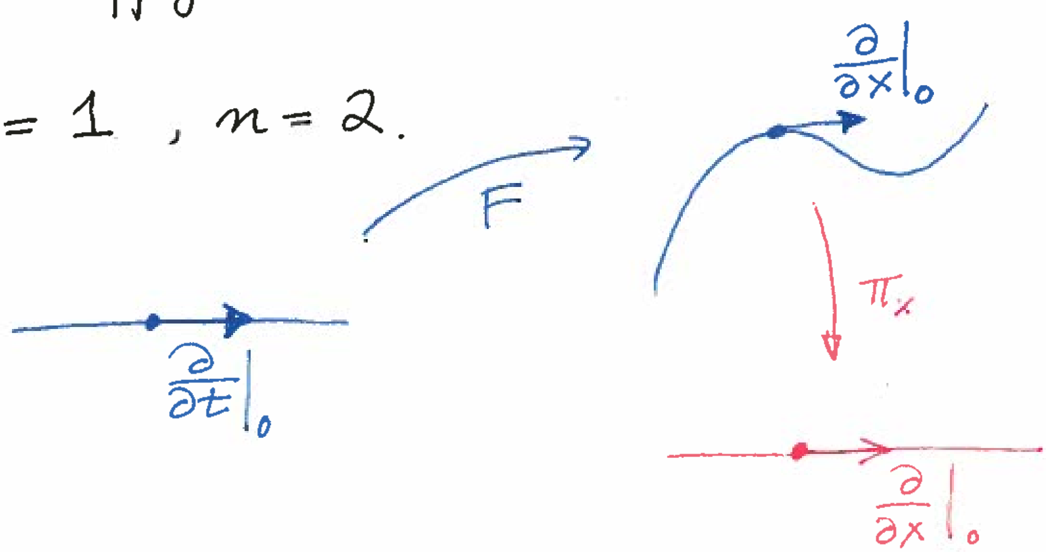
Set $\varphi(x,y) = (x, F(x,y))$. <u>Claim:</u> φ is a diffeo near 0. If so, then $\hat{F}(x,z) = z$ so done.

As

$$d\varphi_0 = \begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x}\Big|_0 & \frac{\partial F}{\partial y}\Big|_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x}\Big|_0 & 1 \end{pmatrix} \leftarrow \begin{array}{c} \det = 1 \\ \text{so} \\ \text{nonsingular} \end{array}$$

we can apply the IVT to do the trick.

(b) $m = 1$, $n = 2$.



Can assume $dF\left(\frac{\partial}{\partial t}\Big|_0\right) = \frac{\partial}{\partial x}\Big|_0$. Now

$d(\pi_x \circ F)_0 \left(\frac{\partial}{\partial t}\Big|_0\right) = \frac{\partial}{\partial x}\Big|_0$ so $\exists\, U \subseteq \mathbb{R}$ containing 0

on which $\pi_x \circ F$ is a diffeo; set $h = (\pi_x \circ F|_U)^{-1}$,

and consider $\psi(x,y) = (h(x), y)$. Then

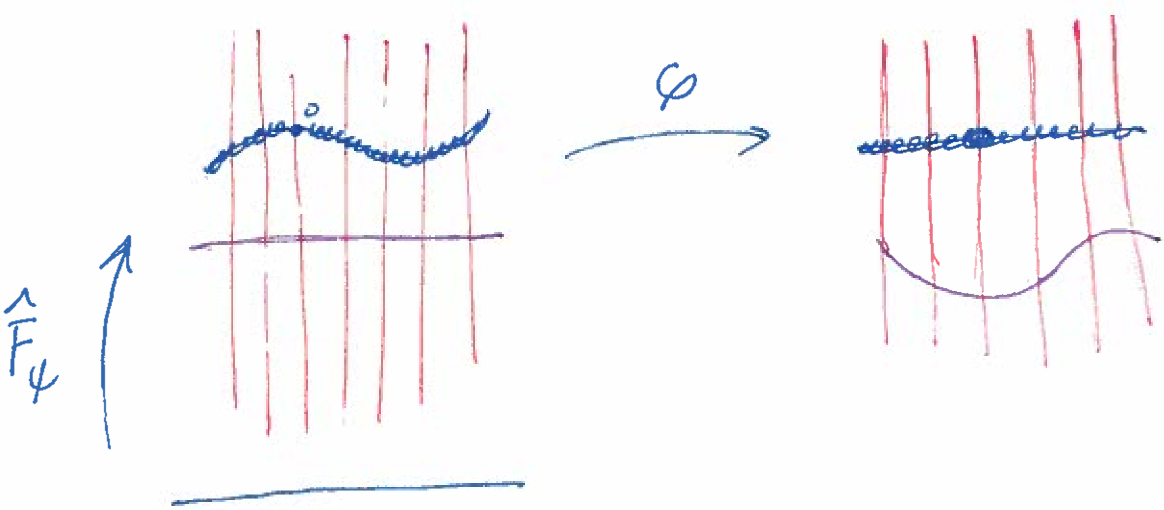
$$\hat{F}_\psi(t) = \big(h(\pi_x(F(t))),\ \pi_y F(t)\big) = (t, \underbrace{\pi_y F(t)}_{f(t)})$$

Set $\varphi(x,y) = \varphi(x, y - f(x))$. Also a

change of coordinates since $d\varphi_0 = \begin{pmatrix} 1 & 0 \\ -f'(x) & 1 \end{pmatrix}$.

Now

$$\hat{F}_{\varphi\circ\psi}(t) = \varphi(\hat{F}_\psi(t)) = (t, f(t) - f(t)) = (t, 0)$$



<u>Thm</u>: Suppose $F: M^m \to N^n$ is a submersion. For any $q \in N$, the preimage $F^{-1}(q)$ is an embedded submanifold of M of dim $m-n$.

<u>Proof Sketch</u>: Set $S = F^{-1}(q)$. [Need to give charts!]

Let $p \in S$. By Thm, $\exists$ chart (U, φ) at p so that

$\varphi(U) = (-1,1)^m$, $\varphi(p) = 0$, and $\hat{F} =$ projection onto first n coor.

So $\varphi(S \cap U) = \underbrace{(0, \dots, 0)}_{n} \times \mathbb{R}^{m-n}$. Define $V = S \cap U$ and

$\psi: V \to \mathbb{R}^{m-n}$ by $\pi_{\text{last } m-n \text{ coor}} \circ \varphi$. Clearly a local homeo.

To check compatibility of charts, use that

$\psi^{-1}: \mathbb{R}^{m-n} \longrightarrow V$ is smooth since its just

$$(x_1, \dots, x_{m-n}) \longmapsto (0, 0, \dots, 0, x_1, \dots, x_{m-n})$$

in the U coordinates. ▨