Lecture 11: Preimages of Regular Values.

Thm: Suppose $F : M^m \to N^n$ is smooth, $p \in M$.

(b) If F is a submersion, $\exists$ charts (U, φ) of M and (V, ψ) of N so that $\varphi(p) = 0$, $\psi(F(p)) = 0$, and

$$\hat{F}(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_n)$$

(a) If F is an immersion, can arrange that

$$\hat{F}(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_m, 0, 0, \dots, 0)$$

Thm: Suppose $F : M \to N$ is a submersion. For any $q \in N$, the preimage $F^{-1}(q)$ is an embedded submfld of dim $=$ dim $M -$ dim N.

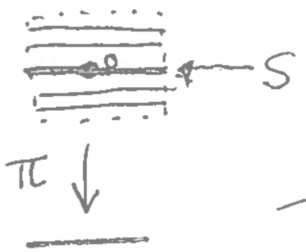
Proof Sketch: Set $S = F^{-1}(q)$. [Need charts!]

Let $p \in S$. By (b), $\exists (U, \varphi)$ so that

$\varphi(p) = 0$, $\hat{F} = \begin{smallmatrix}\text{proj. onto}\\ 1^{st}\ n \text{ coor}\end{smallmatrix}$, and $\varphi(U) = (-1,1)^m$

[See picture on next pg] So $\varphi(S \cap U) = \underbrace{(0, \dots, 0)}_{n} \times (-1,1)^{m-n}$

Now define $V = S \cap U$ and $\psi : V \longrightarrow \mathbb{R}^{m-n}$ by $\pi_{\substack{\text{last}\\ m-n\\ \text{coor}}} \circ \varphi$. Clearly,



ψ is a homeo from V to $(-1,1)^{m-n} = W$

To check compatibility, note that $\psi^{-1} : W \longrightarrow M$ is smooth and that ψ is the restriction of a smooth fn $\overline{\psi} : U \longrightarrow W$, namely $\overline{\psi} = \pi_{\substack{\text{last}\\ m-n}} \circ \varphi$. ▨

__Def:__ $F : M \longrightarrow N$ smooth. A $p \in M$ is a __critical point__ of F if dF_p is __not__ onto. A $q \in N$ is a __critical value__ if it is the image of a critical pt; otherwise, it is a __regular value__

__Ex:__ $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\qquad F(x,y,z) = x^2 + y^2 + z^2$

$$dF_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$$

Critical pts : $(0,0,0)$ $\qquad$ Critical values : 0

Regular values : $\mathbb{R} \setminus \{0\}$.

<u>Thm</u>: Suppose $F : M \to N$ is smooth. If q is a regular value, then $F^{-1}(q)$ is an embedded submanifold of $\dim = \dim M - \dim N$.

<u>Proof</u>: Let $M_s = \{ p \in M \mid p \text{ is } \underline{\text{not}} \text{ a critical pt} \}$.
$\Updownarrow$
dF_p is onto

which is open by cont. of det(submatrix of $d\hat{F}$).

So M_s is itself a smooth mfld and contain all of $F^{-1}(q)$ since q is a reg. value. Now apply the last thm to $F|_{M_s}$ ▨

<u>Sard's Thm</u>: $F : M \to N$ smooth. Then the set of critical values of F has measure 0 in N.
$\Rightarrow$ any open $U \subseteq N$ contains a regular value.

<u>Def</u>: $A \subseteq \mathbb{R}^n$ has measure 0 if $\forall \varepsilon > 0$, there is a countable collection of open boxes of total volume $< \varepsilon$.

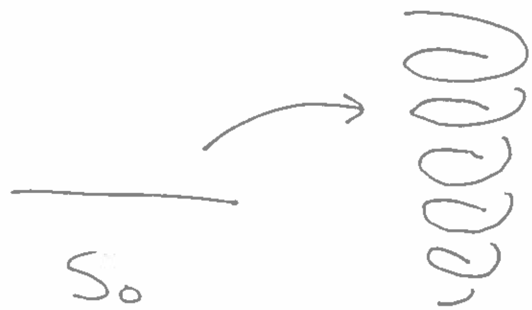
Ex: ① $A = \{pt\} \subseteq \mathbb{R}^n$ 

② 

$\subseteq \mathbb{R}^2$ ③ $\mathbb{R} \subseteq \mathbb{R}^2$ 

Non Ex: ① A open $\subseteq \mathbb{R}^n$ 

Def: M a smooth manifold. Then $A \subseteq M$ has measure 0 if $\forall$ charts (U, φ) the set $\varphi(A \cap U) \subseteq \mathbb{R}^n$ has measure 0.



Suppose $\dim M < \dim N$

Then every $p \in M$ is critical. So

$$\text{regular values} = N \setminus F(M)$$

Cor: If $\dim M < \dim N$, then for any smooth $F: M \to N$ we have ⓐ $F(M)$ has measure 0

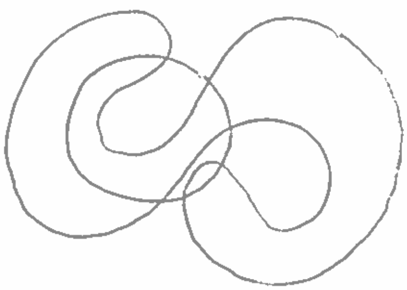
ⓑ F is not onto.

Note: $F(M)$ can still be fairly big, e.g. the HW example of $\mathbb{R} \to T^2$ with dense image.

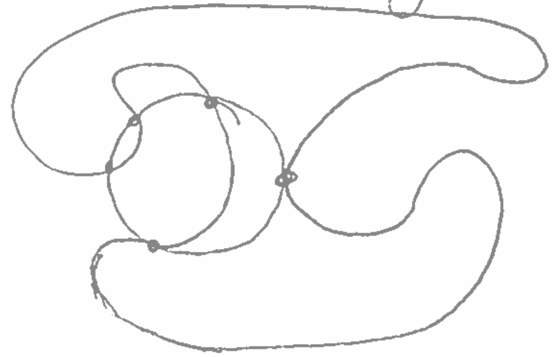
Contrast: $\exists$ continuous onto fn $\mathbb{R} \longrightarrow \mathbb{R}^n$ for every n.

Application: Transversality

Consider $\alpha, \beta : S^1 \to \mathbb{R}^2$ smooth embedding



transverse.



not transverse.

Cor: Any such α, β can be perturbed to smooth embeddings that are transverse.