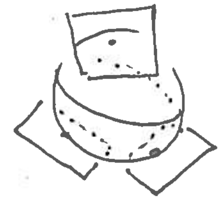
Lecture: Tangent bundles and vector fields

Previously: $p \in M^n$ smooth, have $T_pM \cong \mathbb{R}^n$.



Tangent bundle: M smooth

$$TM = \coprod_{p \in M} T_pM$$

Ex: $M = \mathbb{R}^n$

$$T\mathbb{R}^n = \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

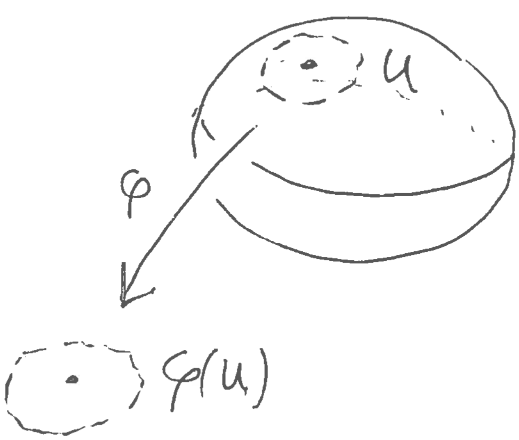
$(x_1, \dots, x_n)$ coor on $\mathbb{R}^n$ $\longmapsto$ $(x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ coor on $\mathbb{R}^{2n}$.

In general, TM is a smooth manifold of $\dim = 2 \dim M$.

If (U, φ) is a chart of M, we get a bijection

$$d\varphi : TU \longrightarrow T\varphi(U) \cong \varphi(U) \times \mathbb{R}^n$$

[Why is this a bijection?]

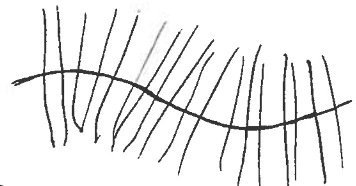


Use $(TU, d\varphi)$ as charts on TM. [Have to check transition fns are smooth.]

Important: In general, $TM \not\cong M \times \mathbb{R}^n$.

While $TS^1 \underset{[HW!]}{\cong} S^1 \times \mathbb{R}$ it turns out that $TS^2 \not\cong S^2 \times \mathbb{R}$

TM is an example of a vector bundle

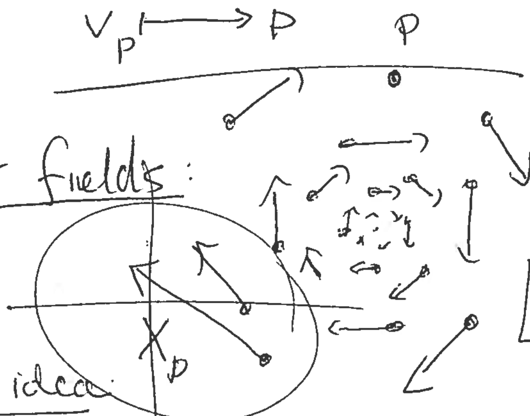


Easy facts: • If $F: M \to N$ is smooth, the diff

$$dF: TM \longrightarrow TN \text{ is a smooth map}$$
$$v_p \longmapsto dF_p(v_p) \in T_{F(p)}N$$

• $\pi: TM \longrightarrow M$ is smooth
$$v_p \longmapsto p$$

Vector fields:



$F(x,y) = (+y, -x)$

[But now want to remember where the vectors are based]

Basic idea:

$$p \longmapsto X_p \in T_pM$$

$$(x,y) \longrightarrow (y,-x) \in T_{(x,y)}\mathbb{R}^2$$

$$\mathbb{R}^2 \xrightarrow{X} T\mathbb{R}^2$$

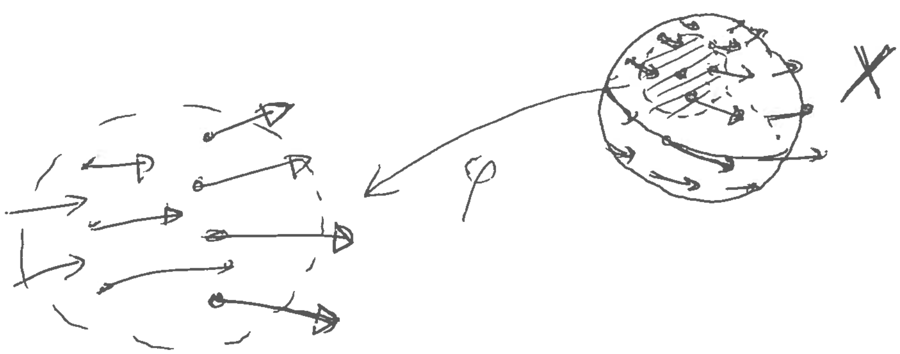
A vector field on M is a fn $X: M \longrightarrow TM$ where $\forall p \in M$ we have $X(p) \in T_pM$.

Equivalently, $X \longrightarrow TM$ satisfies that $\pi \circ X = id_M$

(X is a section of π)

$$\begin{tikzcd} v_p \in TM \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "X"'] \end{tikzcd} \qquad \begin{array}{c} v_p \\ \downarrow \\ p \end{array}$$

The vector field X is smooth if it is a smooth fn $X \longrightarrow TM$. ~~Alternatively X is~~



On $\varphi(U)$ get a vector field

$$\hat{X}(x_1, \dots, x_n) = d\varphi\left(X_{\varphi^{-1}(x_1, \dots, x_n)}\right) \in T_{(x_1, \dots, x_n)}\varphi(U)$$

$$= \sum_{i=1}^{n} \underbrace{a_i(x_1, \dots, x_n)}_{\varphi(U) \to \mathbb{R}} \frac{\partial}{\partial x_i}\Big|_{(x_1, \dots, x_n)}$$

The original X is smooth if the a_i are smooth for all smooth charts.

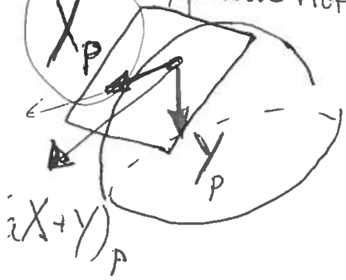
[Vector fields will be a central object in this course.]

$$\mathfrak{X}(M) = \left\{ \begin{array}{l} \text{smooth vector} \\ \text{fields on } M \end{array} \right\}$$

This is an $\mathbb{R}$-vector space: Given

$X, Y \in \mathfrak{X}(M)$, $a \in \mathbb{R}$, ~~set~~ define

explain notation



$$(aX+Y)_p = aX_p + Y_p$$

If $f \in C^\infty(M)$ set $fX \in \mathfrak{X}(M)$

by to be $(fX)_p = f(p)X_p$.

Q: Is $\mathfrak{X}(M)$ a $C^\infty(M)$-vector space? A: No, since $C^\infty(M)$ is not a field, just a ring.] Still $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

Suppose $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Define $Xf \in C^\infty(M)$ by

$$(Xf)(p) = X_p(f)$$

Notice that $X : C^\infty(M) \to C^\infty(M)$ is $\mathbb{R}$-linear and

$$\big(X(f\cdot g)\big)(p) = X_p(f\cdot g) = f(p)(X_p g) + (X_p f)\cdot g(p)$$
$$= f\cdot(Xg) + g\cdot(Xf)$$

That is X is a <u>derivation</u> of <u>$C^\infty(M)$</u>.

[Q: How does this differ from what was used to define T_pM?]

[A: $v_p \in T_pM$ is a linear functional $\big(C^\infty(M) \to \mathbb{R}\big)$, $+$ Leibnitz rule
$X \in \mathfrak{X}(M)$ is an $\mathbb{R}$-linear map $C^\infty(M) \to C^\infty(M)$.

<u>Thm</u> (Lee 8.15) $\mathfrak{X}(M) = \left\{ \begin{matrix} \text{derivations} \\ \text{on } C^\infty \end{matrix} \right\}$

<u>Integral curves</u>:

A path $\gamma : I \to M$ is an <u>integral curve</u> for $X \in \mathfrak{X}(M)$

if $\gamma'(t) = X_{\gamma(t)}$ for all t.

= Existence and uniqueness for solutions to ODE's:

Given $p \in M$, there exists $\varepsilon > 0$ and an integral curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$.

Any two such integral curves agree on the intersection of their domains

Flows: Fix $X \in \mathfrak{X}(M)$. Given $p \in M$,

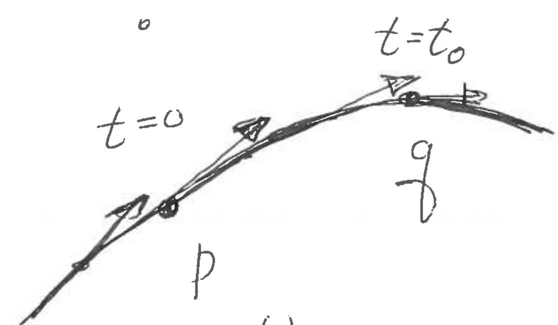
define $\theta^{(p)}: \mathbb{R} \to M$ to be the unique integral curve through p with $\theta^{(p)}(0) = p$.

[Note: Assuming for now integral curve defined for all time.]

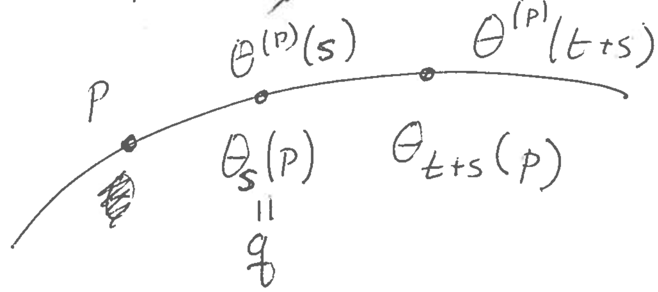
Suppose $\theta^{(p)}(t_0) = q$. Then $\theta^{(q)}(t) = \theta^{(p)}(t + t_0)$

So if we set

$$\theta_t : M \to M$$



by $\theta_t(p) = \theta^{(p)}(t)$



we have that

$$\theta_{t+s}(p) = \theta^{(p)}(t+s) = \theta^{(q)}(t) = \theta_t(q)$$

$$= \theta_t(\theta_s(p))$$

That is: $\boxed{\theta_{t+s} = \theta_t \circ \theta_s}$