Lecture 13: Lie Groups

Vector fields: $(p \in M) \longmapsto X_p \in T_p M$ [Section of the tangent bundle.]

$\mathfrak{X}(M) = \{\text{smooth vector fields}\}$

[Originated in study of sym. of diff eqns, but now...]

Lie group: A smooth mfld G which is also a group and where

$$m : G \times G \longrightarrow G \qquad \text{and} \qquad i : G \longrightarrow G$$
$$(g, h) \longmapsto g \cdot h \qquad\qquad\qquad g \longmapsto g^{-1}$$

are both smooth.

Ex: ① $(\mathbb{R}^n, +)$ $\quad m(v, w) = v + w \quad i(v) = -v$

② $(\mathbb{R}^\times = \mathbb{R} \setminus \{0\}, \cdot)$

③ $\underbrace{GL_n(\mathbb{R}) = \{A \in \overbrace{M_n(\mathbb{R})}^{n \times n \text{ matrices}} \mid \det A \neq 0\}}_{\text{open hence a mfld of dim } n^2} \subseteq \mathbb{R}^{n^2}$

m = matrix multiplication

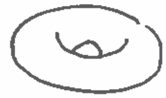
$n=2$:

$$\left.\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \begin{pmatrix} aa' + bc' & \cdot \\ \cdot & \cdot \end{pmatrix} \\ i\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{aligned}\right\} \text{Both smooth}$$

[How are these different than $(\mathbb{R}^n, +)$? What is $GL_1 \mathbb{R}$.]

4. $O(n) = \{A \in M_n(\mathbb{R}) \mid A \cdot A^t = I\}$
   $U(n) = \{A \in M_n(\mathbb{C}) \mid A \cdot \bar{A}^t = I\}$
   $SL_n\mathbb{R} = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$

   skim

5. $S^1 \subseteq \mathbb{C}$ under multiplication, , ...

For fixed $g \in G$ consider the maps $G \longrightarrow G$ given by $L_g : h \longmapsto gh$ and $R_g : h \longrightarrow hg$

left translation — right translation

These are smooth and in fact diffeomorphisms (inverses are $L_{g^{-1}}$ and $R_{g^{-1}}$ respectively.)

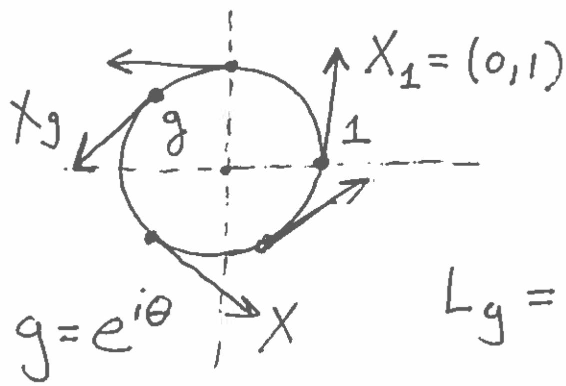
Note $L_{g_2} \circ L_{g_1} = L_{g_2 g_1}$ and $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$

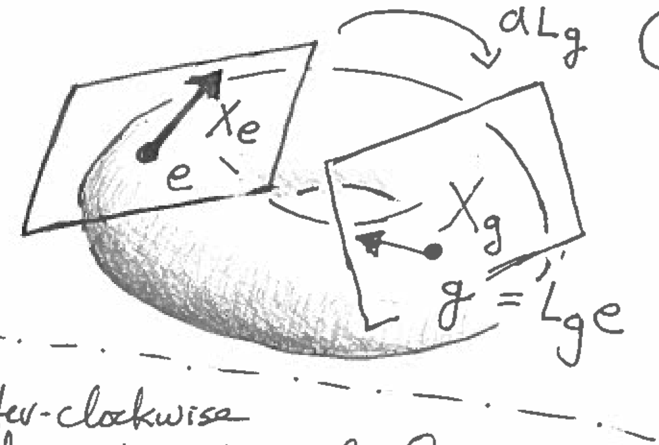
Consider $X_e \in T_e G$. Define a vector field

e ← ident. elt.

on G by $X_g = (dL_g)(X_e)$.

Ex. $G = S^1$





$L_g = R_g =$ counter-clockwise rotation through angle θ.

Lemma: Let G be a Lie group, and choose $X_e \in T_eG$. the vector field $g \longmapsto X_g = (d(L_g))(X_e)$ is smooth.

Proof: Let $\gamma : I \to G$ be a path with $\gamma'(0) = X_e$.

Then $X_g = d(L_g)(X_e) = (L_g \circ \gamma)'(0) = (g \cdot \gamma)'(0)$.

Now consider $(0, X_e) \in T_gG \times T_eG = T_{(g,e)}G \times G$

which is the velocity vector of $\alpha(t) = (g, \gamma(t))$

at $t = 0$. Note that

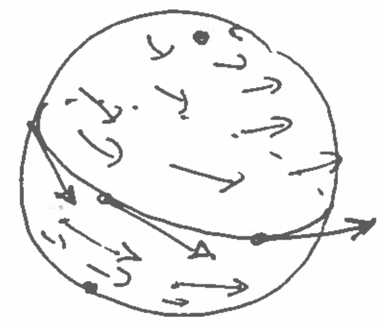
$$dm_{(g,e)}(0, X_e) = (m \circ \alpha)'(0) = (g \cdot \gamma)'(0)$$
$$= X_g$$

Smoothness of X follows from smoothness of

$dm : T(G \times G) \longrightarrow TG.$ $\blacksquare$

Note that if $X_e \neq 0$ so is $d(L_g)(X_e) = X_g$.

That is, G has a smooth nowhere vanishing vector field. (Fact: S^2 can't be made into a Lie group!)



Cor. If G is a Lie gp of dim n, then $TG \cong G \times \mathbb{R}^n$.

Sketch: Let $X_e^1, \dots, X_e^n$ be basis for T_eG. Define vector fields X^k by $X_g^k = dL_g(X_e^k)$. Then

$$\begin{aligned} G \times \mathbb{R}^n &\longrightarrow TG \\ (g, (t_1, \dots, t_n)) &\longmapsto \sum_{k=1}^{n} t_k X_g^k \in T_g G \end{aligned}$$

is a diffeomorphism.

Suppose $F : M \longrightarrow N$ is a diffeomorphism.

For $X \in \mathfrak{X}(M)$ the <u>push</u>-<u>forward</u> $F_* X \in \mathfrak{X}(Y)$ is given by

$$(F_* X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$$

$$= (dF^{-1})_q^{-1}\left(X_{F^{-1}(q)}\right)$$

A vector field $X \in \mathfrak{X}(\overset{\text{Lie gp}}{G})$ is left invariant if $(L_g)_* X = X$ for all $g \in G$.

Note that if X is left-invariant then

$$X_g = \left((L_g)_* X\right)_g = (dL_g)_{\underset{(L_g)^{-1}(g)}{e}} X_e$$

which is the vector field we defined above. In fact, any such vector field is left invariant.

<u>Cor</u>: $\underbrace{\{X \in \mathfrak{X}(G) \mid \text{left-invariant}\}}_{\text{Lie Algebra}} \cong T_e G$