Lecture 14: More on Lie Groups.

<u>Lie Group</u>: Smooth mfld with a group structure whose operations are smooth.

For $g \in G$ have <u>left translation</u> $L_g : G \longrightarrow G$
$$h \longmapsto g \cdot h$$

<u>Thm</u>: Given $X_e \in T_eG$, the vector field $g \longmapsto (dL_g)(X_e)$ is smooth.

Suppose $F : M \longrightarrow N$ is a diffeomorphism. The <u>push-forward</u> of $X \in \mathfrak{X}(M)$ is $F_* X$ defined by

$$(F_* X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$$

An $X \in \mathfrak{X}(\text{Lie gp } G)$ is <u>left invariant</u> if

$$(L_g)_* X = X \quad \text{for all } g \in G.$$

For any such X, one has

$$X_g = \left((L_g)_* X\right)_g = (dL_g)_e(X_e)$$

which has the above form. Conversely, if X

has this form it is left invariant since:

$$((L_g)_* X)_h = (dL_g)_{g^{-1}h}(X_{g^{-1}h})$$

$$= (dL_g)_{g^{-1}h}(dL_{g^{-1}h})(X_e)$$

$$= d(L_g \circ L_{g^{-1}h})(X_e) = (dL_h)(X_e)$$

$$= X_h.$$

Cor: $\underbrace{\{X \in \mathfrak{X}(G) \mid X \text{ left invariant}\}}_{\text{Lie Algebra}} \cong T_eG$

[Will put a mult. on this that makes it into a non-associative ring. Can use to classify Lie groups.]

Def: G, H Lie gps. A Lie group homomorphism is group homomorphism $F: G \to H$ which is also a smooth map. F is a Lie group isomorphism if it is also a diffeomorphism.

Ex: (a) $S^1 \hookrightarrow \mathbb{C}^\times = GL_1\mathbb{C}$

(b) $(\mathbb{R},+) \longrightarrow (\mathbb{R}^\times, \times)$

$t \longmapsto \exp(t)$

image is $\mathbb{R}^+$ and

$$(\mathbb{R},+) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}^+, \times)$$

are isomorphic.

(c) $\pi : \mathbb{R} \longrightarrow S^1$

$t \longrightarrow e^{2\pi i t}$

[What was this an example of?]

(d) $\det : GL_n\mathbb{R} \longrightarrow \mathbb{R}^\times$

Def: $F: M \to N$ smooth has constant rank r if $\forall p \in M$ the dimension of $dF_p(T_pM)$ is r.

$r = \dim M$ is a immersion.

$r = \dim N$ is a submersion.

Thm: A Lie group homomorphism $F: G \to H$ has constant rank.

Cor: $SL_n\mathbb{R} = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$ is a Lie group.

Pf of Cor: $SL_n\mathbb{R} = \det^{-1}(1)$ so enough to show that det is a submersion, i.e. the const rank $\neq 0$.

Consider $\gamma(t) = \begin{pmatrix} t & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$. Then $d(\det)(\gamma'(1))$

$= (\det \circ \gamma)'(1) = 1 \neq 0$ so rank $= 1$ ∎

<u>Cor</u>: A Lie group homomorphism is an isomorphism iff it is bijective.

<u>Pf</u>: If $F: G \to H$ is a bijective Lie gp homomorphism, then it must have a regular value (Sard) and by Thm in fact F must be a submerison. Since F is bijective, must have $\dim G = \dim H$ and hence dF_g an isomorphism for all g. F is thus a diffeo by the Invese Fn Thm. ∎

<u>Proof of Thm</u>: Let $g \in G$. <u>Claim</u>: $\operatorname{rank}(dF_g) = \operatorname{rank}(dF_e)$

Consider:

$$\begin{tikzcd} T_g G \arrow[r, "dF"] & T_{F(g)} H \\ T_e G \arrow[u, "\cong\ d(L_g)"] \arrow[r, "dF"'] & T_e H \arrow[u, "d(L_{F(g)})\ \cong"'] \end{tikzcd}$$

Now $F \circ L_g = L_{F(g)} \circ F$ since

$$F \circ L_g(g') = F(gg') = F(g)F(g') = L_{F(g)}F(g')$$

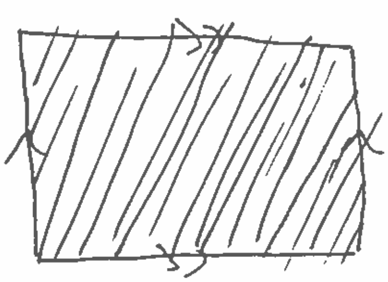
Hence, the chain rule says above commutes and in particular dF_g and dF_e have the same rank. ▨

Def: Suppose $F : H \longrightarrow G$ is an injective Lie gp homomorphism. The subgp $F(H) \leq G$ is called a Lie subgroup of G.

Note: Such an F is an immersion, so $F(H)$ is an immersed submfld.

Ex: $G = S^1 \times S^1$ $H = \mathbb{R}$ $\alpha \in \mathbb{R}$ irrational

$$F(t) = (e^{2\pi i t}, e^{2\pi i \alpha t}) : H \longrightarrow G$$



Note: $F(H)$ is not an embedded submfld.