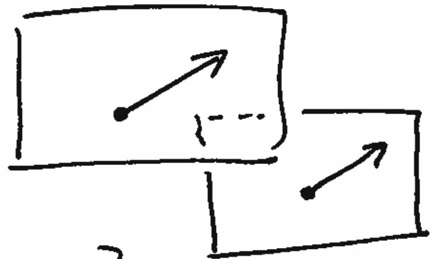
Lecture 5: Derivations as tangent vectors.

Goal: Define tangent spaces, derivatives of maps between manifolds.



Last time: $T_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n$

$C^\infty(\mathbb{R}^n) = \{\text{smooth fns } \mathbb{R}^n \longrightarrow \mathbb{R}\}$

$\mathcal{D}_a = \left\{ \begin{array}{l} \text{set of derivations} \\ \text{at } a \in \mathbb{R}^n \text{, i.e.} \\ w : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R} \end{array} \quad \begin{array}{l} ① \; w(cf+g) = c\,w(f) + w(g) \\ \\ ② \; w(fg) = w(f)g(a) + f(a)w(g) \end{array} \right\}$

Ex: $v_a \in T_a\mathbb{R}^n$ ~~gives the~~ has a directional derivative $D_{v_a} : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$.

Prop: The map $\begin{array}{ccc} T_a\mathbb{R}^n & \longrightarrow & \mathcal{D}_a \\ v_a & \longmapsto & D_{v_a} \end{array}$ is an isomorphism of $\mathbb{R}$-vector spaces.

[Think about for $n = 1$.]

Pf: 1 This is a linear transformation because

$$D_{v_a + c w_a} f = (D_a f)(v_a + c w_a)$$
$$= (D_a f)(v_a) + c (D_a f)(w_a)$$
$$= D_{v_a} f + c D_{w_a} f.$$

2 1-1: Suppose $v_a \longmapsto \begin{pmatrix} \text{const derivation} \\ 0 \end{pmatrix}$. If x_i is the i^{th} coordinate fn, then for any v_a we have

$$D_{v_a} x_i = (\nabla x_i) \cdot v_a = e_i \cdot v_a = v_i .$$

So if $D_{v_a} f = 0$ for all f, must have all $v_i = 0$, i.e. $v_a = 0$. So the kernel is trivial and the map 1-1.

onto: Let $w \in \mathcal{D}_a$. Set $v_i = w(x_i)$ and consider the corresp. $v \in T_a \mathbb{R}^n$. Claim: $D_v = w$.

Suppose $f \in C^\infty(\mathbb{R}^n)$

$$f(x) = f(a) + (D_a f)(x-a) + \text{Error}$$

$$= f(a) + \sum_i \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$+ \underbrace{\sum_{i,j}(x_i-a_i)(x_j-a_j)\underbrace{E_{ij}(x)}_{\text{smooth fn}}}_{\text{vanishes at } a.}$$

$$E_{ij}(x) = \frac{1}{2}\int_0^1 (1-t)^2 \frac{\partial^2}{\partial x_i \partial x_j} f(a+t(x-a$$

Note $w(1) = w(1\cdot 1) = 1\cdot w(1) + w(1)\cdot 1$
$= 2\,w(1). \implies w(\text{const}) = 0.$

If g and h both vanish at a, then

$$w(g\cdot h) = w(g)\,h(a) + g(a)\cdot w(h) = 0.$$

Thus

$$w(f) = 0 + \sum_i \frac{\partial f}{\partial x_i}(a)\,w(x_i) + \sum_{i,j} 0$$

$$= \sum_i \frac{\partial f}{\partial x_i}(a)\,v_i = D_v f.$$

So $w = D_v$. ▨

M^n smooth.

$C^\infty(M) = \{\text{smooth } M \to \mathbb{R}\}$ — $\mathbb{R}$-vector space, ring, algebra (over $\mathbb{R}$).

<u>Define:</u>

$$T_pM = \left\{ \begin{array}{l} w: C^\infty(M) \to \mathbb{R} \text{ which are} \\ \bullet\ \mathbb{R}\text{-linear} \\ \bullet\ w(f\cdot g) = w(f)\, g(p) + f(p)\, w(g) \end{array} \right\}$$

[Prop shows this new def agrees with original one for $M = \mathbb{R}^n$]

[So now have a vector space at each point, will show soon that it has the right dimension.]

Suppose $F: M \to N$ is smooth. The <u>derivative/differential</u> of F at $p \in M$

$$dF_p : T_pM \longrightarrow T_{F(p)}N$$

is defined by

in $C^\infty(N)$

$$dF_p(v)(g) = v(g \circ F)$$

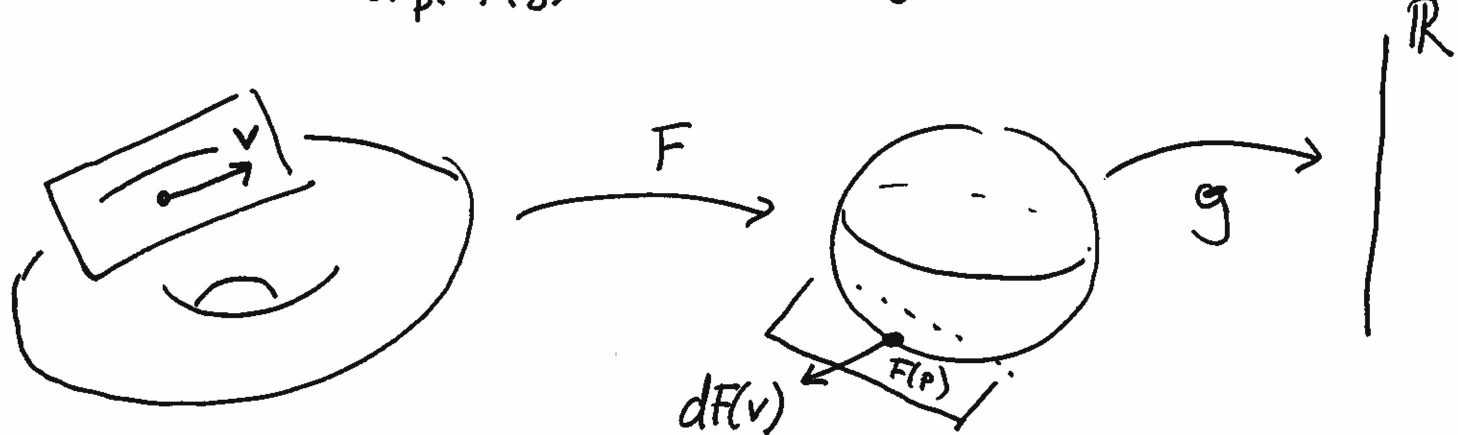
Derivation!

<u>Check</u> (Skip in class): $dF_p(v)$ is a derivation at $F(p)$, so in $T_{F(p)}N$.

$$(dF_p(v))(g\cdot h) = v\big((g\circ F)\cdot(h\circ F)\big)$$

$$= v(g\circ F)\,(h\circ F(p)) + (g\circ F)(p)\;v(h\circ F)$$

$$= dF_p(v)(g)\;h(F(p)) + g(F(p))\;dF_p(v)(h)$$



<u>Props:</u> $M \xrightarrow{F} N \xrightarrow{G} P$ smooth, $p \in M$.

- dF_p is linear.
- $d(G\circ F)_p = dG_{F(p)} \circ dF_p$
- $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$
- If F is a diffeomorphism, then dF_p is an isomorphism with $(dF_p)^{-1} = dF_p^{-1}$.

[Pf: one-liners from the definition.]

Tangent vectors are local: $v \in T_pM$.

If $f, g \in C^\infty(M)$ agree on a nbhd U of p, then $v(f) = v(g)$.

Proof: By next HW, there is $\psi \in C^\infty(M)$ which is 0 at p and 1 outside U.

Set $h = f - g$. Since h is 0 on U,

$h = \psi \cdot h$. Hence

$$v(f) - v(g) = v(h) = v(\psi \cdot h) = 0$$

since both ψ and h vanish at p. ∎

Lemma: Suppose $p \in U \underset{\text{open}}{\subseteq} M$. If $i : U \to M$ is inclusion, then $di_p : T_pU \to T_pM$ is an isomorphism.

Pf: By preceeding, can determine some $v \in T_pM$ by what it does to smooth functions supported inside U.

Prop: M^n smooth, $p \in M$. Then $\dim T_pM = n$.

Proof: Let (U, φ) be a smooth chart around p. Then $\varphi|_U : U \to \varphi(U)$ is a diffeomorphism, so $d(\varphi|_U)_p$ is an isom of

$$\begin{array}{ccc} T_pU & \xrightarrow[\cong]{} & T_{\varphi(p)}\varphi(U) \\ \rotatebox{90}{$\cong$} & & \rotatebox{90}{$\cong$} \\ T_pM & & \underbrace{T_p\mathbb{R}^n}_{\text{has dim } n!} \end{array}$$

$\blacksquare$