Lecture 6: More on the tangent space.

Last time:

$$T_pM = \left\{ \begin{array}{l} v: C^\infty(M) \longrightarrow \mathbb{R} \text{ sat } \forall f, g \in C^\infty(M), c \in \mathbb{R} \\ \cdot\ v(f + cg) = v(f) + c\,v(g) \\ \cdot\ v(fg) = v(f)g(p) + f(p)v(g) \end{array} \right.$$

If $f: M \to N$ is smooth, get $dF_p : T_pM \longrightarrow T_{F(p)}N$ ← linear transformation

by $dF_p(v)(g \in C^\infty(N)) = v(g \circ F)$

Locallity: If $f, g \in C^\infty(M)$ agree on a nbhd of p then $v(f) = v(g)$ for all $v \in T_pM$.

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Lemma: $U \subseteq M$ an open subset of a smooth mfld. For every $p \in U$, the inclusion $i: U \to M$ gives an isomorphism $di_p : T_pU \to T_pM$.

Cor: $\dim T_pM = \dim M$.

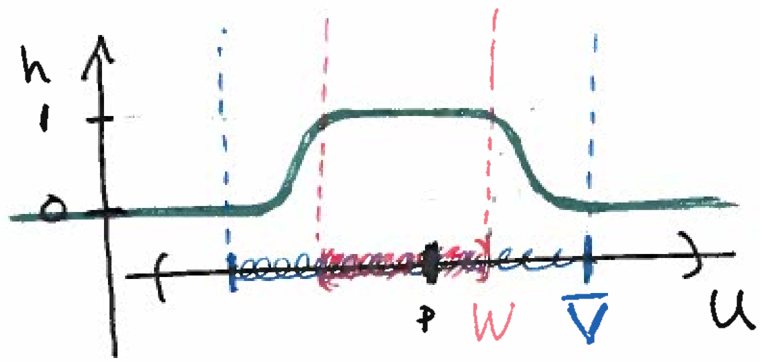
Pf of Cor: Let (U, φ) be a smooth chart at p.

Then $\varphi : U \to \varphi(U)$ is a diffeomorphism. So

$$\begin{array}{ccc} T_pU & \xrightarrow[\cong]{d\varphi_p} & T_p\,\varphi(U) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ T_pM & & T_p\mathbb{R}^n \leftarrow \text{has dim } n. \end{array}$$

$\blacksquare$

Pf of Lemma: By HW, there are open nbhds $W \subseteq V \subseteq U$ of p with $\overline{V} \subseteq U$ and a smooth $h : M \to \mathbb{R}$ where $h = 1$ on W and $h = 0$ outside of V.



By locality, if $f \in C^\infty(M)$ then $v(h \cdot f) = v(f)$ for $\forall v \in T_pM$. Thus $v \in T_pM$ is determined by its values on

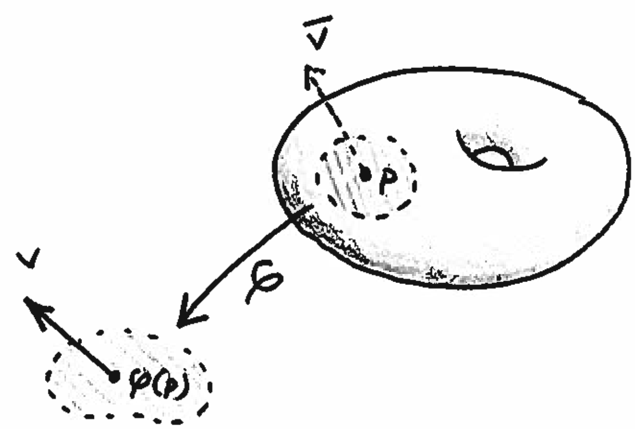
$$\{ f \in C^\infty(M) \mid f \text{ vanishes outside } V \}$$

and any derivation on $\uparrow$ gives an elt of T_pM.

The same is true for U, i.e. can ident $T_p U$ with derivations on $\{ f \in C^\infty(U) \mid f \text{ vanishes outside } V \}$.

Since these two sets of fns are equal (just extend any $f \in C^\infty(U)$ vanishing outside V by 0 outside U) we get $T_p U \cong T_p M$. ▨

<u>Other</u> <u>points</u> of <u>view</u>:



1 $(p, U, \varphi, v) = \bar{v}$

$\bar{v} \in T_p M$ is $d\varphi^{-1}_{\varphi(p)} v$

~~since def~~ since $d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}\varphi(U) \longrightarrow T_p M$

$$T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n$$

<u>Local coordinates</u>: $\mathbb{R}^n$ with coordinates $x_1, x_2, \dots, x_n$

Standard basis for $T_a\mathbb{R}^n = \{e_1, e_2, \dots, e_n\}$

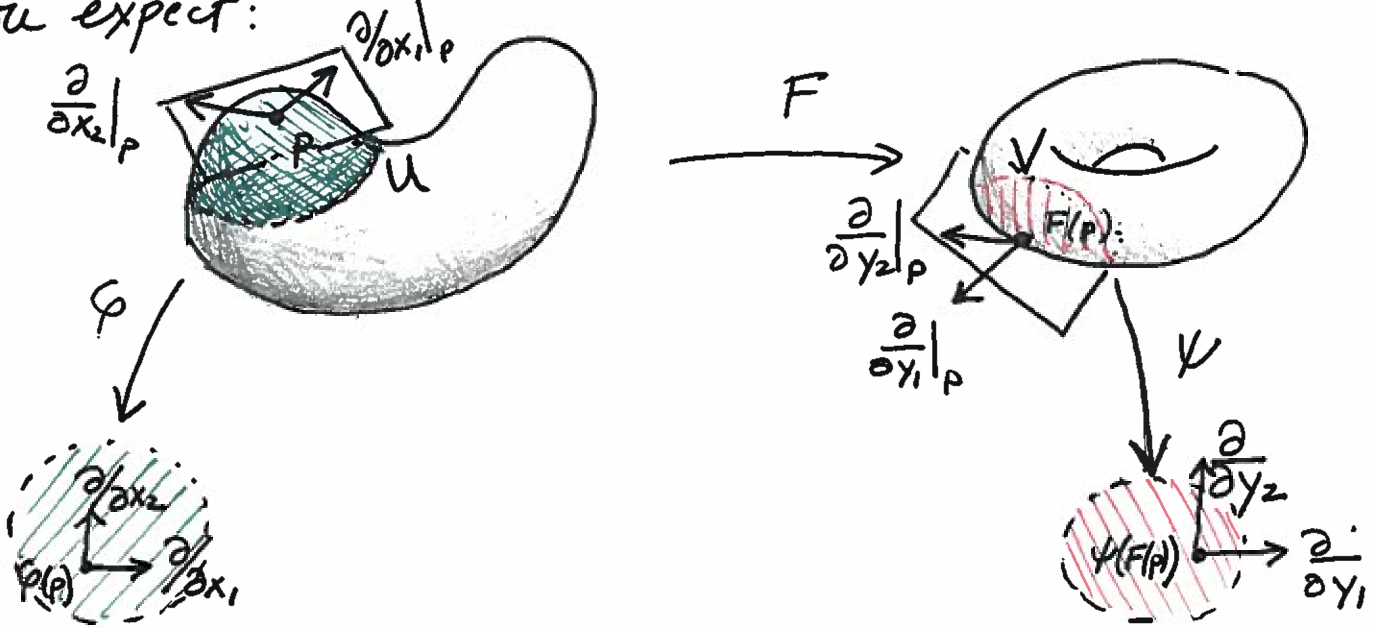
$e_i = (0, \dots, 1, \dots 0)$ ← ith place

Since also think of $T_a\mathbb{R}^n$ as directional derivatives/derivations often denote e_i as $\frac{\partial}{\partial x_i}$. Given (U, φ),

define $\left.\frac{\partial}{\partial x_i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right) = d(\varphi^{-1})_p\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right)$

to get a basis for T_pM. These work like

you expect:



Matrix of dF_p with respect to $\left.\frac{\partial}{\partial x_i}\right|_p$ and $\left.\frac{\partial}{\partial y_i}\right|_{F(p)}$

is $D_{\varphi(p)}(\psi\circ F\circ\varphi^{-1})$.

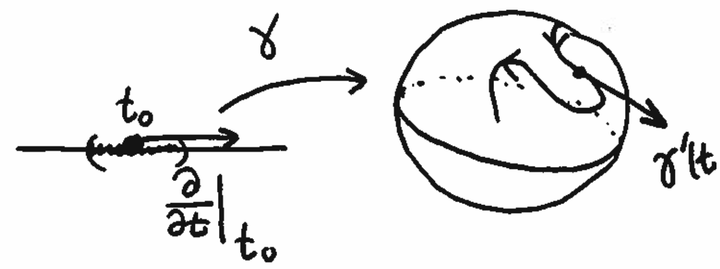
② Germs. [Likely skip refer to text.] pg. 71.

$$C_p^\infty(M) = \left\{(U, f) \;\middle|\; \begin{matrix} f \in C^\infty(U) \\ p \in U \end{matrix}\right\} \Big/ (U,f)\sim(V,g)$$

if $\exists$ open nbhd W of p where $f = g$.

Can work with derivations on this instead.

3 <u>Curves</u>: A smooth $\gamma : J \to M$ where $J \subseteq \mathbb{R}$ is an interval. The <u>velocity</u> of γ at time t_0 is



$$\gamma'(t_0) = d\gamma\left(\frac{\partial}{\partial t}\Big|_{t_0}\right) \in T_{\gamma(t_0)} M$$

<u>Props</u>: [Check on your own.]

(a) If $f \in C^\infty(M)$, then $\gamma'(t_0)(f) = (\underbrace{f \circ \gamma}_{J \to \mathbb{R}})'(t_0)$

(b) If $F : M \to N$ is smooth, then

$$dF_{\gamma(t_0)}(\gamma'(t_0)) = (F \circ \gamma)'(t_0)$$

(c) ~~Any~~ Every $v \in T_pM$ is $\gamma'(t_0)$ for some $\gamma : J \to M$.

$$\mathcal{V}_p M = \left\{ \begin{matrix} \text{smooth curves} \\ \gamma : J \to \mathbb{R} \end{matrix} \;\middle|\; \gamma(0) = p \right\} \Big/ \gamma_1 \sim \gamma_2 \text{ if } \forall f \in C^\infty(M) \text{ we have } (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

Problem 3-8 (on HW #3) shows $\mathcal{V}_pM \cong T_pM$.