Math 416: HW 11 due Wednesday, May 4, 2016.

Important note: This assignment is due on **Wednesday** not Friday.

More important note: This is the last homework assignment of the semester!

Most important note: There will be a combined final exam for sections B13 and C13 of Math 416, which will be held on Friday, May 6 from 1:30-4:30 in Psychology 23. Please notify me immediately if you have another exam in that timeslot.

Webpage: http://dunfield.info/416

Office hours: Here is my schedule for the rest of the semester:

- Friday, April 29: 11am–noon.
- Monday, May 2: 3:30–4:30pm.
- Tuesday, May 3: 3:30–4:30pm.
- Thursday, May 5: noon–1:30pm, 3:30–4:30pm.
- Friday, May 6: 9–10am.

Problems:

1. Suppose V is a finite-dimensional inner product space and T a linear operator on V. Prove that $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$ and $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.

2. Let T be a *normal* operator on a finite-dimensional inner product space V.

   (a) Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$.

   (b) Prove that the subspaces $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are orthogonal.

   (c) Give an example of a (non-normal) linear operator S where $\mathcal{N}(S) \neq \mathcal{N}(S^*)$ and $\mathcal{R}(S) \neq \mathcal{R}(S^*)$.

   Hint: Problem 1 is your friend here.

3. A matrix $A \in M_{n\times n}(\mathbb{R})$ is *Gramian* if there is a $B \in M_{n\times n}(\mathbb{R})$ such that $A = B^t B$. Prove that A is Gramian if and only if A is symmetric and all of its eigenvalues are non-negative.

   Hint: For $(\Longleftarrow)$, note that A is diagonalizable via an orthonormal basis $\{u_1, \ldots, u_n\}$ where u_i is an eigenvector of A with eigenvalue λ_i. Consider the linear operator T on $\mathbb{R}^n$ where $T(u_i) = \sqrt{\lambda_i}\, u_i$. Now take $B = [T]_{\mathrm{std}}$ and check that $A = B^t B$.

4. Section 6.5 of [FIS], Problem 11.

5. Section 6.5 of [FIS], Problem 17.

6. Section 6.5 of [FIS], Problem 24.

7. Suppose $A \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix with $\det(A) = 1$. (Recall from a prior assignment that any orthogonal matrix has determinant ± 1.) In this problem, you will show L_A is rotation about a line W in $\mathbb{R}^3$, where W passes through the origin.

   (a) First, show that any (real) eigenvalue of A must be ± 1.

(b) Note that A has at least one eigenvalue since its characteristic polynomial $f(t)$ has odd degree and hence at least one real root λ. In this step, you'll show that 1 is always an eigenvalue. If instead $\lambda = -1$, then $f(t) = (-1-t)(t^2 + bt + c)$ for some $b, c \in \mathbb{R}$. Use that $\det(A) = 1$ to prove that $c < 0$ and hence by the quadratic formula that $f(t)$ splits completely over $\mathbb{R}$. Now show that the eigenvalues of A are -1 and 1, with algebraic multiplicies 2 and 1 respectively.

(c) Let v_1 be an eigenvector for A with eigenvalue 1, and set $W = \text{span}(\{v_1\})$. Prove that L_A preserves $W^\perp$ and acts on it by an orthogonal transformation.

(d) Use Theorem 6.23 of the text to argue that the action of L_A on $W^\perp$ is by a rotation. Hint: If instead the restriction was a reflection, find a basis of $\mathbb{R}^3$ consisting of eigenvectors for A which shows instead that $\det(A) = -1$.

8. Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ and let P be the parallelepiped spanned by them. Consider the matrix $G \in M_{n\times n}(\mathbb{R})$ where $G_{ij} = \langle v_i, v_j \rangle$. (As usual, the inner product here is just the ordinary dot product.)

   (a) Show that G is Gramian.

   (b) Show that $\det(G) \geq 0$.

   (c) Show that the unsigned volume of P is $\sqrt{\det(G)}$.

   In fact, G is usually called the Gram matrix of these vectors.