Lecture 32: More on inner products (§6.1 and 6.2)

Last time: V a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An <u>inner product</u> is a $\langle\cdot,\cdot\rangle : V^2 \longrightarrow \mathbb{F}$ satisfying

a) $\langle x+z, y\rangle = \langle x,y\rangle + \langle z,y\rangle$

b) $\langle cx, y\rangle = c\langle x,y\rangle$

c) $\langle y, x\rangle = \overline{\langle x,y\rangle}$

d) $\langle x,x\rangle$ is in $\mathbb{R}_{>0}$ for $x \neq 0$.

for all $x,y,z \in V$ and $c \in \mathbb{F}$.

From $\langle\ ,\ \rangle$, we define $\|x\| = \sqrt{\langle x,x\rangle}$ for $x \in V$ and call it the <u>norm/length</u> of x.

Thm: Suppose V is an inner product space. For all $x,y \in V$ and scalar c one has:

a) $\|cx\| = |c|\,\|x\|$

b) $\|x\| \iff x = 0$

c) $|\langle x,y\rangle| \leq \|x\|\,\|y\|$ (Cauchy Schwarz)

d) $\|x+y\| \leq \|x\| + \|y\|$ (Δ-inequality)

Proof: Parts (a) and (b) are easy; will do (c) and (d) when the field of scalars is $\mathbb{R}$.

c) If $y = 0$ get $0 \leq 0$ as needed, so assume $y \neq 0$. Notice that if we scale y by $c \in \mathbb{R}$, then both sides of (c) change by $|c|$ as per part (a). Replacing y with $\frac{1}{\|y\|} y$ we can thus assume that $\|y\| = 1$. Now

$$0 \leq \|x - \langle x,y\rangle y\|^2 = \langle x,x\rangle - 2\langle x,y\rangle^2 + \langle x,y\rangle^2 \overbrace{\langle y,y\rangle}^{1}$$
$$= \|x\|^2 - \langle x,y\rangle^2$$

and so $|\langle x,y\rangle| \leq \|x\|$ as needed.

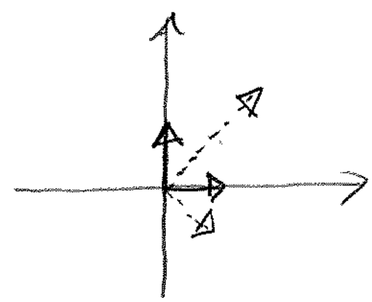
d) $$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$
$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(by (c))}.$$
$$= (\|x\| + \|y\|)^2$$

$\blacksquare$

<u>Def</u>: Suppose V is an inner product space.

Vectors $x, y \in V$ are <u>orthogonal/perpendicular</u> when $\langle x, y \rangle = 0$. A subset $S \subseteq V$ is <u>orthogonal</u> when each pair of distinct vectors is orthogonal. A vector $x \in V$ is unit when $\|x\| = 1$. Finally, a subset of V is <u>orthonormal</u> when it is orthogonal and consists entirely of unit vectors.

<u>Ex</u>: $\mathbb{R}^2$, dot product



- $S = \{e_1, e_2\}$ is orthonormal
- $S = \{2e_1 + 2e_2, e_1 - e_2\}$ is orthogonal.
- $S = \{e_1, e_1 + e_2\}$ is <u>not</u> orthogonal.
- $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$ is orthonormal

<u>Def</u>: An <u>orthonormal basis</u> for an inner product space V is a basis which is orthonormal.

<u>Thm</u>: An finite dim'l inner product space has an orthonormal basis.

[Will prove this next time...]

Thm: Suppose $S = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal subset of an inner product space V. If $y \in \text{span}(S)$, then $y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$.

Ex: ($\mathbb{R}^3$, dot product) $S = \{e_1, e_2, e_3\}$

$y = (4, 3, -2) = 4e_1 + 3e_2 - 2e_3$

$\langle y, e_1 \rangle = 4 \quad \langle y, e_2 \rangle = 3 \quad \langle y, e_3 \rangle = -2$.

Proof: As $y \in \text{span}(S)$, there are scalars a_i with $y = \sum_{i=1}^{k} a_i v_i$. Then $\langle y, v_j \rangle = \sum_{i=1}^{k} \langle a_i v_i, v_j \rangle$

$$= \sum_{i=1}^{k} a_i \underbrace{\langle v_i, v_j \rangle}_{\substack{= 0 \text{ when } i \neq j \\ = \|v_j\|^2 = 1 \text{ when } i = j}} = a_j \text{ for all } 1 \leq j \leq k.$$

Thus $\sum_{i=1}^{k} \langle y, v_i \rangle v_i = \sum_{i=1}^{k} a_i v_i = y$ as needed $\blacksquare$

Cor: Suppose $S = \{v_1, \ldots, v_k\}$ is othogonal with all $v_i \neq 0$. If $y \in \text{span}(S)$ then $y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$

Proof: Set $\bar{v}_i = \frac{v_i}{\|v_i\|}$ so that $\bar{S} = \{\bar{v}_1, \dots, \bar{v}_k\}$ is orthonormal. As $\text{span}(\bar{S}) = \text{span}(S)$, given $y \in \text{span}(S)$ we have

$$y = \sum_{i=1}^{k} \underbrace{\langle y, \bar{v}_i \rangle}_{\frac{1}{\|v_i\|}\langle y, v_i \rangle} \bar{v}_i = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

$\blacksquare$

Cor: Suppose $S \subseteq V$ is orthogonal and $0 \notin S$. Then S is linearly independent.

Pf: Since scaling vectors by non-zero amounts does not change dependence, it suffices to show that any orthonormal $v_1, \dots, v_k$ are linearly indep. If $\sum_{i=1}^{k} a_i v_i = 0$, then as before we have

$$0 = \langle v_j, 0 \rangle = \langle v_j, \sum_{i=1}^{k} a_i v_i \rangle$$
$$= a_j$$

for each j. Thus $v_1, \dots, v_k$ is linearly indep as needed.

$\blacksquare$

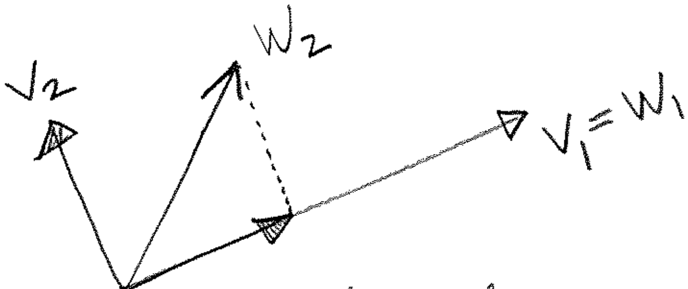
Gram-Schmidt Process:

Suppose $\{w_1, w_2, \dots, w_n\} \subseteq V$ is linearly independent

Set
$$v_1 = w_1$$
$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$
$$\vdots$$
$$v_k = w_k - \sum_{j=0}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$$

<u>Thm</u>: $\{v_1, \dots, v_n\}$ is orthogonal with the same span as $\{w_1, w_2, \dots, w_n\}$.

[Proof next time.]



projection of w_2 onto v_1

Underlying Geometry: