Lecture 37: Diagonalizing self-adjoint operators. (§6.4)

Last time: A linear operator T on an inner product space V is self-adjoint when $T = T^*$. A square matrix is self-adjoint if $A = A^*$.

Synonyms: Hermitian (field $= \mathbb{C}$), symmetric (field $= \mathbb{R}$).

Ex: Suppose W is a subspace of a finite dim'l inner product space V. Then orthogonal projection $T = \mathrm{proj}_W : V \to V$ is self-adjoint.

Proof: Suppose $y_1, y_2 \in V$ and write $y_i = w_i + z_i$ where $w_i = T(y_i) \in W$ and $z_i \in W^\perp$. Then

$$\langle T(y_1), y_2 \rangle = \langle w_1, w_2 + z_2 \rangle =$$

$$\langle w_1, w_2 \rangle + \langle w_1, z_2 \rangle = \langle w_1, w_2 \rangle$$

$$= \langle w_1 + z_1, w_2 \rangle = \langle y_1, T(y_2) \rangle.$$

Thus T is self-adjoint. $\blacksquare$

<u>Note</u>: $R(\mathrm{proj}_W) = W$ and $\mathcal{N}(\mathrm{proj}_W) = W^\perp$.

[In particular, $R^\perp = \mathcal{N}$. On HW, will show this is a general property of normal operators.] On to today's goal...

<u>Thm</u>: Suppose T is a self-adjoint operator on a finite-dim'l inner product space V. Then V has an orthonormal basis β consisting of eigenvectors for T.

<u>Cor</u>: If $A \in M_{n\times n}(\mathbb{R} \text{ or } \mathbb{C})$ then A is diagonalizable.

<u>Lemma</u> (last time) Any eigenvalue of a self-adjoint T is real.

<u>Lemma</u>: Any self-adjoint operator T on $V \neq \{0\}$ has an eigenvector.

<u>Proof</u>: Let β be an orthonormal basis for V and set $A = [T]_\beta$. Note it suffices to show A has an eigenvector. If the field is $\mathbb{C}$, we're done as all matrices in $M_{n\times n}(\mathbb{C})$ have an eigenvector (reason: the char poly must have at least one root). So assume the field is $\mathbb{R}$, in which case $A = A^* = A^t$. It suffices to show that the char poly $f(t)$ has a real root. Now $f(t)$ has at least one root λ in $\mathbb{C}$, which means that $L_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ has an eigenvector with eigenvalue λ. As L_A is self-adjoint (as $A^* = A$), we have $\lambda \in \mathbb{R}$ by the previous lemma. So $f(t)$ has a real root as needed. $\blacksquare$

Proof of theorem: We induct on $\dim(V)$.

If $\dim(V) = 1$, the lemma gives eigenvector v for T. As $v \neq 0$, we have $\beta = \left\{ \frac{v}{\|v\|} \right\}$ is the orthonormal basis of eigenvectors we seek.

Now assume true when $\dim(V) \leq n-1$, and consider V of dim n. Let v_1 be a unit eigenvector of T given by the lemma.

Consider $W = \{v_1\}^{\perp}$.

Claim: $T(W) \subseteq W$

Reason: Suppose $w \in W$. Then $\langle T(w), v_1 \rangle$
$= \langle w, T^*(v_1) \rangle = \langle w, \lambda_1 v_1 \rangle = \overline{\lambda_1} \langle w, v_1 \rangle$
$= 0$ since $w \in \{v_1\}^{\perp}$. Thus $T(w)$ is also in W.

Now W is also an inner product space (with the inner prod. inherited from V), and the restricted linear op $T_W : W \to W$ is still self adjoint. By induction, there is an orthonormal basis $\{v_2, \dots, v_n\}$ of W consisting of eigenvectors of T_W. Then

$$\beta = \{v_1, v_2, \dots, v_n\}$$

is an orthonormal set (since $\langle v_1, v_i \rangle = 0$ for $i > 1$ as $v_i \in \{v_1\}^\perp$) of eigenvectors of T.

As any orthonormal set is linearly indep. and $\dim V = n$, we have shown β is a basis as needed. $\blacksquare$

<u>Next time</u>: Orthogonal and unitary operators. (§6.5)