# Lecture 38: Orthogonal and unitary operators (§6.5)

Convention: Today, $V$ will always be a finite-dim'l inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Story so far: $T$ linear operator on $V$.

Normal: $T \circ T^* = T^* \circ T$

Fact: When $\mathbb{F} = \mathbb{C}$, any normal $T$ is diagonalizable via an orthonormal basis.

Self-adjoint: $T = T^* \Rightarrow$ diagonalizable via an orthornormal basis even if $\mathbb{F} = \mathbb{R}$.

Today: Linear op $T$ on $V$ where

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in V.$$

[Reflect on theme of preserving structure...]

Such $T$ are called orthogonal when $\mathbb{F} = \mathbb{R}$
unitary when $\mathbb{F} = \mathbb{C}$

I'll use the generic term isometry for either field

Ex: $(\mathbb{R}^2, \text{dot})$: rotations, reflections.
$(\mathbb{R}^n, \text{dot})$: rigid motions fixing $0$.

Non Ex: Anything with $N(T) \neq \{0\}$.

<u>Thm</u>: For a linear op $T$ on $V$, the following are equivalent:

a) $T$ is an isometry.

b) $T \circ T^* = T^* \circ T = I_V$ ($\Rightarrow T$ is normal.)

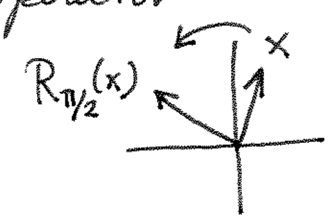
c) For every orthonormal basis $\beta$ of $V$, the image $T(\beta)$ is <u>also</u> an orthonormal basis.

d) For <u>some</u> orthonormal basis $\beta$, $T(\beta)$ is orthonormal.

e) $\|T(x)\| = \|x\|$ for all $x \in V$.

<u>Lemma</u>: Suppose $S$ is a self-adjoint operator on $V$. If $\langle S(x), x\rangle = 0$ for all $x \in V$, then $S = T_0$, that is $S(x) = 0$ for all $x \in V$.

<u>Note</u>: $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ is a normal operator where $\langle x, R_{\pi/2}(x)\rangle = 0$ for all $x \in \mathbb{R}^2$.




<u>Proof of lemma</u>: Pick $\{v_1, \ldots, v_n\}$ an othonormal basis of eigenvectors for $S$. If $S(v_i) = \lambda_i v_i$

then $0 = \langle S(v_i), v_i \rangle = \langle \lambda_i v_i, v_i \rangle = \lambda_i \|v_i\|^2$

and so all $\lambda_i = 0$. Thus $S(v_i) = 0$ for all $i$ and

hence $S = T_0$. $\blacksquare$

<u>Proof of Thm:</u>

<u>(a) $\Rightarrow$ (b):</u> By last time, as $T^* \circ T$ is self-adjoint, so is $S = T^* \circ T - I_V$. For any $x \in V$ we have

$$\begin{aligned}\langle S(x), x \rangle &= \langle T^* \circ T(x), x \rangle - \langle I_V(x), x \rangle \\ &= \langle T(x), T(x) \rangle - \langle x, x \rangle \\ &= 0 \text{ since } T \text{ is an isometry.}\end{aligned}$$

By the lemma, $S = T_0$ and so $T^* \circ T = I_V$.

Then $T^* = T^{-1}$ and so $T \circ T^* = T \circ T^{-1} = I_V$ as well, proving (b).

<u>(b) $\Rightarrow$ (c):</u> Suppose $\beta = \{u_1, \dots, u_n\}$, and set $w_i = T(u_i)$. Now

$$\begin{aligned}\langle w_i, w_j \rangle &= \langle T(u_i), T(u_j) \rangle = \langle u_i, T^* \circ T(u_j) \rangle \\ &= \langle u_i, I_V(u_j) \rangle = \langle u_i, u_j \rangle\end{aligned}$$

Hence $\{w_1, \dots, w_n\}$ is an orthonormal basis, as needed for (c).

(c) $\Rightarrow$ (d): Clear.

(d) $\Rightarrow$ (e): Suppose $\beta = \{u_1, \dots, u_n\}$, $w_i = T(u_i)$, with $\beta$ and $\gamma = \{w_1, \dots, w_n\}$ orthonormal. Write $x = a_1 u_1 + \cdots + a_n u_n$. Then $\|x\|^2 = \langle x, x \rangle$

$$= \sum_{i,j}^{n} a_i \overline{a_j} \langle u_i, u_j \rangle = \sum_{i=1}^{n} |a_i|^2$$ and

similarly $\|T(x)\|^2 = \| \sum a_i w_i \|^2 = \sum_{i=1}^{n} |a_i|^2$.

Thus $\|T(x)\| = \|x\|$ as needed.

(e) $\Rightarrow$ (a): By expanding $\langle x+y, x+y \rangle$, get that

real part

$$Re(\langle x, y \rangle) = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

and if $\mathbb{F} = \mathbb{C}$ then

$$Im(\langle x, y \rangle) = -\frac{1}{2}\left(\|ix + y\|^2 - \|x\|^2 - \|y\|^2\right)$$

imaginary part

Thus $\langle \, , \, \rangle$ is actually determined by the assoc. norm $\|\cdot\|$, and so as $T$ preserves the latter it preserves the former. $\blacksquare$

If a linear op $T$ of $V$ is an isometry and $\beta$ is orthonormal, then by (6) we have

$$I_n = [I_V]_\beta = [T^* \circ T] = [T^*]_\beta [T]_\beta$$
$$= ([T]_\beta)^* [T]_\beta$$

Def: A square matrix $A$ is unitary if $A^* A = I$.
It is orthogonal if $A^t A = I$.

So the matrix of an isometry is always unitary, and when $\mathbb{F} = \mathbb{R}$ it is also orthogonal.

Thm: Suppose $A \in M_{n \times n}(\mathbb{R})$ is orthogonal.
Then $L_A$ is an isometry of $(\mathbb{R}^n, \text{dot})$.

Note: Analog true for $A \in M_{n \times n}(\mathbb{C})$ that are unitary.

Proof: Let $\beta = \{e_1, \dots, e_n\}$ and set $a_i = A e_i$ $= i^{th}$ col of $A$. Then $A^t A = \begin{pmatrix} - a_1 - \\ \vdots \\ - a_n - \end{pmatrix} \begin{pmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{pmatrix}$

$= G$ where $G_{ij} = \langle a_i, a_j \rangle$.

Thus $\gamma = \{a_1, \dots, a_n\} = L_A(\beta)$ is an orthonormal basis for $\mathbb{R}^n$, and so $L_A$ is an isometry by the first thm.

<u>Cor</u>: $A \in M_{n \times n}(\mathbb{R})$. The following are equivalent

i) $A$ is orthogonal.

ii) $A^t = A^{-1}$

iii) The columns of $A$ are an orthonormal basis for $\mathbb{R}^n$

iv) The rows of $A$ ——————— " ———

<u>Pf</u>: Exercise.

<u>Restated Thm</u>: Suppose $A \in M_{n \times n}(\mathbb{R})$ is symmetric. Then there is an orthogonal matrix $Q$ with $Q^t A Q = Q^{-1} A Q$ diagonal.