Lecture 10: Basis and dimension, part II

[FIS §1.6]

Last time: V vector space, $S \subseteq V$.

S spans V: $\mathrm{span}(S) = V$.

S is a basis for V: S is linearly indep. and spans V.

Today: Will show:

Thm A: If V has a finite basis, then any two bases for V have the same number of vectors.

Thm B: Suppose W is a subspace of a finite dim'l vector space V. Then $\dim W \leq \dim V$. Moreover, $\dim W = \dim V$ if and only if $W = V$.

Replacement Thm: Suppose $S = \{s_1, s_2, \ldots, s_n\}$ spans V, and $U = \{u_1, u_2, \ldots, u_k\} \subseteq V$ is linearly independent. Then $k \leq n$ and there is a subset $T \subseteq S$ of size $n-k$ such that $U \cup S$ spans V.

[Explain name. Mention notion implies that the s_i and u_j are distinct. Befor proving this, let's derive Thm A from it.]

Pf of Thm A: Suppose β_1 is a finite basis for V, and β_2 any basis. Set $n = \#\beta_1$. If β_2 was infinite, it would contain a subset $\mathcal{U}$ of $k = n+1$ linearly indep. vectors. As $\operatorname{span}(\beta_1)$ is V, this would violate the Replacement Thm; thus β_2 must be finite. Set $k = \#\beta_2$.

The Rep. Thm with $S = \beta_1$ and $\mathcal{U} = \beta_2$ gives us that $k \leq n$. With $S = \beta_2$ and $\mathcal{U} = \beta_1$, it implies $n \leq k$. So $n = k$ as needed. ▨

Proof of Repl. Thm: Can assume $n > 0$. Will induct on k.

Base case: $k = 0$, i.e. $\mathcal{U} = \emptyset$. Then $k \leq n$ and we just take $T = S$.

<u>Inductive step</u>: Suppose $\{u_1, \dots, u_{k+1}\}$ is linearly indep. and we know $k \leq n$ and that

$$I = \{u_1, \dots, u_k\} \cup \{s_1, \dots, s_{n-k}\} \text{ spans } V.$$

[Note: implicitly relabeled the s_i]

<u>Must show</u>: $k+1 \leq n$ and can replace one s_i with u_{k+1} and still span.

As I spans V, there are scalars where

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k + b_1 s_1 + \cdots + b_{n-k} s_{n-k}.$$

Can't have all $b_i = 0$ or we would get a linear dependence among the u_i. Relabel so that $b_{n-k} \neq 0$. In particular $n-k \geq 1$ $\Longrightarrow k+1 \leq n$ as needed. Also, dividing through by b_{n-k} we get that

$$s_{n-k} \in \text{span} \underbrace{\{u_1, \dots, u_k, s_1, \dots, s_{n-k-1}\}}_{J}$$

So

$$\operatorname{span}(J) \overset{\text{HW!}}{=} \operatorname{span}(J \cup \{s_{n-k}\}) \supseteq \operatorname{span}(I) = V$$

Thus $\operatorname{span}(J) = V$ and we have successfully replaced one s_i by u_{k+1}, completing the induction. ▨

<u>Proof of Thm B</u>: Let β_V be a finite basis for V. If W has a finite basis β_W then by the Repl. Thm

number of elts

ⓐ $\# \beta_W \leq \# \beta_V$

ⓑ If $\# \beta_W = \# \beta_V$, then $\beta_W \cup \emptyset$ spans V, that is, $W = V$.

<u>Finding β_W</u>: If $W = \{0\}$ there's nothing to prove. Otherwise, pick $w_1 \neq 0$ in W.

If $\operatorname{span}(w_1) = W$, then $\{w_1\}$ is our basis.

Otherwise, pick $w_2 \in W$ which is not in $\text{span}(w_1)$. Note that $\{w_1, w_2\}$ is linearly independent. Repeating, we construct $\{w_1, \dots, w_k\} \subseteq W$ which is linearly indep, and can increase the size by one unless $\text{span}\{w_1, \dots, w_k\} = W$.

By the replacement thm, $k \leq \# \beta_V$, so we eventually build a finite basis for W. This completes the proof of Thm B. $\blacksquare$

<u>Cor</u>: Suppose $\dim V = n < \infty$. Then

a) Any linearly indep subset of V has size $\leq n$.

b) For a subset S of size n, the following are equivalent:
   i) S is a basis
   ii) S is linearly indep
   iii) S spans V.

Pf: a) is direct from the Repl. Thm

b) Have $ii) \overset{\text{Last}}{\underset{\text{time}}{\Longrightarrow}} i) \overset{\text{Def.}}{\Longrightarrow} iii)$. So it remains to show $iii) \Rightarrow ii)$. If S were linearly dependent, could span V with $< n$ vectors $\Rightarrow \dim V < n$. So S must be linearly independent.

Cor: Suppose W is a subspace of a finite dim'l V. Any basis for W can be extended to one for V.