Lecture 8: Linear dependence and independence

[§1.5 of FIS]

Suppose W is a subspace of a vector space V.
How do we answer questions such as:

1. Given $w_1, w_2, \ldots, w_k$ in W, is $\text{span}(\{w_1, \ldots, w_k\})$ all of W?

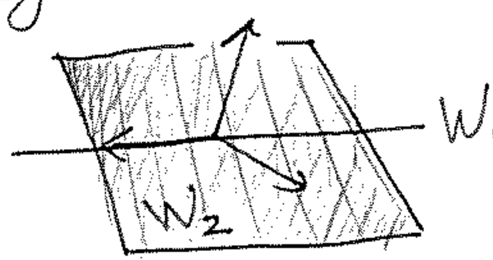
2. What is smallest number of w_i needed span W? How can we find such vectors?

3. Given v in V, is v in W?

4. In $\mathbb{R}^3$, we've seen 4 kinds of subspaces:

   i) $\{0\}$ ii) $\mathbb{R}^3$ iii) lines through 0

   iv) planes through 0.

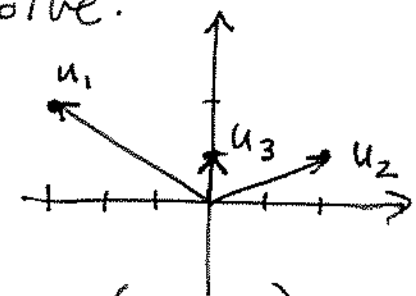


How do we distinguish these mathematically and generalize to $\mathbb{R}^n$?

Vectors $u_1, \ldots, u_k$ in V are <u>linearly dependent</u> if there are scalars $a_1, a_2, \ldots, a_k$, <u>not all 0</u>, such that $a_1 u_1 + \cdots + a_k u_k = 0$.

<u>Ex</u>: $V = \mathbb{R}^2$ $\quad u_1 = (-3, 2)$ $\quad u_2 = (2, 1)$ $\quad u_3 = (0, 1)$

Are these dependent? Trying to solve:



$$a_1 u_1 + a_2 u_2 + a_3 u_3 =$$
$$(-3a_1 + 2a_2,\ 2a_1 + a_2 + a_3) = (0, 0)$$

Gives 2 equations
$$\begin{aligned} -3a_1 + 2a_2 &= 0 \\ 2a_1 + a_2 + a_3 &= 0 \end{aligned}$$

So we're trying to find the <u>null space</u> of $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. Turns out, this is $\{(2t, 3t, -7t) \mid t \in \mathbb{R}\}$. So these are dependent.

A linear dependency means there is a redundancy amongst the vectors in terms of the span: $2u_1 + 3u_2 - 7u_3 = 0$

$$\Rightarrow u_1 = -\frac{3}{2}u_2 + \frac{7}{2}u_3$$

and so $\text{span}(u_1, u_2, u_3) = \text{span}(u_2, u_3)$

(see prob 1 on HW 3).

In contrast $a_2 u_2 + a_3 u_3 = 0$ has only the trivial solution [easy from earlier eqn's] and so u_1, u_2 are called <u>linearly independent</u>.

<u>Ex</u>: $V = \mathbb{R}^3$ $\qquad W = \text{span}(\{u_1, u_2, u_3\})$

$u_1 = (-1, 1, 2)$

$u_2 = (1, 2, 1)$

$u_3 = (5, 1, -4)$

<u>Q</u>: What is W geometrically?

[Should be a plane or $\mathbb{R}^3$.]

Are these linearly dependent? The condition

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$$

gives rise to eqns

$$\begin{aligned} -a_1 + a_2 + 5a_3 &= 0 \\ a_1 + 2a_2 + a_3 &= 0 \\ 2a_1 + a_2 - 4a_3 &= 0 \end{aligned}$$

Equivalently, we need to find the null space of

$$A = \begin{pmatrix} -1 & 1 & 5 \\ 1 & 2 & 1 \\ 2 & 1 & -4 \end{pmatrix} = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$$

useful shortcut!

which row reduces to

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and so the sol'ns are $\{(3t, -2t, t) \mid t \in \mathbb{R}\}$

Taking $t = 1$, get

$$u_3 = -3u_1 + 2u_2$$

More generally, we say $S \subseteq V$ is "subset" *linearly dependent* if there are distinct vectors $u_1, \ldots, u_k$ in S which are linearly dependent; otherwise call S *linearly independent*.

Notes: ① The set $\{0\}$ is linearly dependent, as $a \cdot 0 = 0$ for all scalars a.

② If $u \neq 0$, then $\{u\}$ is linearly independent. as $au = 0$ for $a \neq 0$ gives $u = \frac{1}{a}(au) = \frac{1}{a}0 = 0$.

③ The empty set ϕ is linearly independent.

Aside: By convention $\mathrm{span}(\phi) = \{0\}$

<u>Thm</u> Suppose $u_1, \ldots, u_k$ are nonzero vectors in V, and consider the subsp $W = \text{span}(\{u_i\})$. Then there exists a linearly independent subset $u_{i_1}, \ldots, u_{i_\ell}$ of the u_i where $\text{span}(\{u_{i_1}, \ldots, u_{i_\ell}\})$ is all of W.

<u>Pf</u>: Suppose the $\{u_i\}$ are linearly dependent. Then we can express one of the u_i as a linear combination of the others. Reindexing the u_i if necessary, we can arrange that

$$u_k = a_1 u_1 + \cdots + a_{k-1} u_{k-1}$$

In particular, $W = \text{span}(\{u_1, \ldots, u_{k-1}\})$

Repeating this argument, we eventually arrive at $\underbrace{u_1, \ldots, u_\ell}_{\text{relabelled many times!}}$ which are linearly independent and which span V. ▨

<u>Note</u>: Always end up with at least one u_{i_1}.