Lecture 9: Basis and dimension, part I.

Last time: A subset S of a vector space V is linearly dependent if there are distinct $u_1, u_2, \ldots, u_k$ in S and scalars $a_1, \ldots, a_k$, not all zero, such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_k u_k = 0.$$

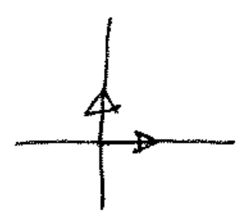
Otherwise, call S linearly independent. [Restate]

Def: A subset S of a vector space V spans (generates) V if $\operatorname{span}(S) = V$.

[Also, say that S spans V; makes sense for subspaces.]

Def: A basis for V is a subset β which is linearly independent and spans V.

Ex: (1) $\beta = \{(1,0), (0,1)\}$ is a basis for $\mathbb{R}^2$



(2) Set $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0), \ldots,$ $e_n = (0,\ldots,0,1)$. Then $\beta = \{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$.

[Query] These are linearly independent as if $0 = \sum_{i=1}^{n} a_i e_i = (a_1, a_2, .., a_n)$ we have all $a_i = 0$.

Ex: $P_n(\mathbb{R}) = \left\{ \begin{matrix} \text{polys of} \\ \deg \leq n \end{matrix} \right\}$ has basis $\{1, x, x^2, .., x^n\}$.

Thm: If a vector space V has a finite spanning set S then V has a finite basis.

Pf. Suppose $S = \{u_1, .., u_n\}$. By last time, there is a linearly indep. subset $\{u_{i_1}, .., u_{i_k}\}$ which has the same span as S, and this is a basis for V. $\blacksquare$

Fact: Any vector space has a basis, even $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Thm: Suppose $\beta = \{u_1, \ldots, u_n\}$ is a subset of a vector space V. Then β is a basis if and only if every v in V can be uniquely expressed as a linear comb of the vectors in β.

Proof: ($\Longrightarrow$) Suppose β is a basis. As β spans V, given any $v \in V$ there are scalars a_i with

$$v = a_1 u_1 + \cdots + a_n u_n$$

To show uniqueness, suppose that

$$v = b_1 u_1 + \cdots + b_n u_n$$

Subtracting, we get

$$0 = (a_1 - b_1) u_1 + (a_2 - b_2) u_2 + \cdots + (a_n - b_n) u_n$$

As the u_i are linearly independent, must have $a_i = b_i$ for all i. Thus any v has a unique expression as a linear comb of the u_i.

($\Leftarrow$) Suppose every v in V has a unique expression in terms of β. In particular, $\operatorname{span}(\beta) = V$. To show linear indep, suppose

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$$

As $0 \in V$ has only one expression in terms of β, we must have all $a_i = 0$. So β is linearly indep and hence a basis. $\boxed{}$

<u>Thm</u>: If V has a finite basis, then any two bases for V have the same number of vectors.

<u>Pf</u>: Next time.

<u>Def</u>: A vector sp V is <u>finite-dimensional</u> if has a finite basis. The number of vectors in any basis is called the <u>dimension</u> of V and denoted $\dim V$.

Otherwise, call V infinite dimensional.

Ex: $\dim \mathbb{R}^n = n$ $\qquad$ $\dim\{0\} = 0$

$\dim P_n(\mathbb{R}) = n+1$

$\dim\left(P(\mathbb{R}) = \text{polys in } x \text{ any degree}\right) = \infty$

as basis is $\{1, x, x^2, x^3, \dots\}$

Thm: Suppose W is a subspace of a finite dim'l V. Then $\dim W \leq \dim V$. Moreover $\dim W = \dim V$ if and only if $W = V$.

Pf: Next time.

Corollary: If $\beta = \{u_1, \dots, u_n\}$ is linearly independent in V where $\dim V = n$, then β is a basis for V.

Proof: Let $W = \operatorname{span}(\beta)$. As β spans W and is linearly indep, it is a basis for W.

So $\dim W = n = \dim V$ and so $V = W$ by thm. Thus β spans V and so is a basis for V. $\blacksquare$

<u>Ex</u>: $\beta = \left\{ \begin{array}{l} u_1 = (1,2,3) \\ u_2 = (4,5,6) \\ u_3 = (7,8,10) \end{array} \right\}$ is a basis for $\mathbb{R}^3$.

By Cor. need only check linear indep.
From last time, we know this is equiv to the null space of $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$ being $\{0\}$.

This is the case as this row reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponding to $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$.