Lecture 23: Properties of the determinant [§3.1 and §4.3] ①

Last time: $A \in M_{n\times n}(\mathbb{R})$.

① $A \xrightarrow{R_r \leftrightarrow R_s} B \Rightarrow \det(B) = -\det(A)$

② $A \xrightarrow{cR_r} B \Rightarrow \det(B) = c\det(A)$

③ $A \xrightarrow{cR_s + R_r} B \Rightarrow \det(B) = \det(A)$

Today: $\det(AB) = \det(A)\det(B)$

[Strategy: Relate row ops to matrix multiplication, but first here's one more easy consequence of what we learned last time.]

Recall $\text{rank}(A) = \dim(\text{ColSp}(A)) \overset{\text{Thm}}{=} \dim(\text{RowSp}(A))$.

Thm: For $A \in M_{n\times n}(\mathbb{R})$, if $\underbrace{\text{rank}(A) < n}_{\Leftrightarrow A \text{ \underline{not} invertible}}$ then $\det(A) = 0$.

Proof: As $\text{rank}(A) < n$, some row is a linear combination of the others, say

$$a_r = c_1 a_1 + \cdots + c_{r-1}a_{r-1} + c_{r+1}a_{r+1} + \cdots + c_n a_n$$

where a_i is row i of A. If we do row ops

$-c_i R_i + R_r$ for $i = 1, \dots, n$ and $i \neq r$, then we get a matrix B whose r^{th} row is 0.

Hence by last time $\det(B) = 0$. By 3, we have $\det(A) = \det(B)$ so $\det(A) = 0$ as req'd. $\blacksquare$

__Def__: An $n \times n$ elementary matrix is the result of doing a single row operation to I_n.

__Ex__: 1 $I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_4 \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

2 $I_4 \xrightarrow{5R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3 $I_3 \xrightarrow{5R_2 + R_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_4 \xrightarrow{-3R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$

Note that these are the identity matrix except in at most 1 place (2 and 3) or 4 places (1).

<u>Thm</u>: Suppose E is the elementary matrix where $I_n \xrightarrow{R} E$. If $A \in M_{n\times n}(\mathbb{R})$ then

$$A \xrightarrow{R} EA.$$

<u>Ex</u>: $R_1 \leftrightarrow R_2$, $n=2$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{E}\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$3R_1$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix}$$

$-R_1 + R_2$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

<u>Proof of Thm</u>: HW #7.

<u>Thm</u>: Every elementary matrix is invertible.

<u>Proof</u>: Suppose $I_n \xrightarrow{R} E$. Let R' be the row op that reverses R, that is $A \xrightarrow{R} B \xrightarrow{R'} A$ for all $A \in M_{n\times n}(\mathbb{R})$. [Query: why does this exist?]

Let E' be the elementary matrix associated with R'. By previous theorem, have

$E'E$ = result of doing R' to $E = I_n$

EE' = result of doing R to $E' = I_n$.

So E is invertible with inverse E'. $\square$

<u>Thm</u>: $A \in M_{n \times n}(\mathbb{R})$ is invertible if and only if it is the product of elementary matrices.

<u>Proof</u>: ($\Leftarrow$) If $A = E_1 E_2 \cdots E_\ell$ with E_k elementary, then each E_k is invertible and so

$$A^{-1} = E_\ell^{-1} E_{\ell-1}^{-1} \cdots E^{-1}$$

by HW.

($\Rightarrow$) If A is invertible, then

$$B = (A \mid I_n) \xrightarrow[\text{ops}]{\text{row}} (I_n \mid A^{-1}) = C$$

As each row op can be implemented by mult. by an element matrix, we have E_k where

$$E_\ell \cdots E_2 E_1 B = C$$

which implies

$$E_\ell E_{\ell-1} \cdots E_2 E_1 A = I_n$$

and so

$$A = E_\ell^{-1} \cdots E_2^{-1} E_1^{-1}$$

As the E_k^{-1} are also elementary, we're done. $\blacksquare$

<u>Thm</u>: $\det(AB) = \det(A)\det(B)$.

<u>Proof</u>: If $\operatorname{rank}(AB) < n$ then $\det(AB) = 0$. Moreover, one of A, B must have rank $< n$ and so one of $\det(A)$, $\det(B)$ is 0. So $\det(AB) = \det(A)\det(B)$ in this case.

So have reduced to the case where A, B, and AB all have rank n. In particular

$$A = E_1 \cdots E_\ell \qquad B = E_{\ell+1} \cdots E_m$$

where E_k are elementary. The result now follows from

Claim: Suppose $C = E_1' \cdots E_p'$ where E_k' are elementary. Then

$$\det(C) = (-1)^{(\#\text{ of type } ① \ E_k')} \cdot \begin{pmatrix} \text{product of } c_k \\ \text{in all type ② } E_k' \end{pmatrix}$$

Proof of Claim: C is obtained from I_n, which has det 1, by the row ops $R_p', R_{p-1}', \dots, R_1'$.

By last time, only the type ① and ② ops change the det and moreover do so in a way that proves the claim. ▨

Cor: For $A \in M_{n \times n}(\mathbb{R})$, have $\det(A) \neq 0$ if and only if A is invertible.

Proof: If A is invertible, then $1 = \det(I_n)$
$= \det(A \cdot A^{-1}) = \det(A)\det(A^{-1}) \Rightarrow \det(A) \neq 0$.

If instead A is not invertible, then $\det(A) = 0$ by first result of today. ▨