Lecture 18: Matrices: Inverses and rank.

Last time: An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B with $AB = BA = I_n$.

When A is invertible, the inverse is unique and written as A^{-1}.

Thm: ~~Suppose~~ $A \in M_{n \times n}(\mathbb{R})$. The following are equivalent: ① A is invertible

② The linear transformation $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto Ax$ is invertible.

③ The nullspace $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is $\{0\}$.

Proof: First, we show ① $\Longleftrightarrow$ ②. If ①, then $L_A \circ L_{A^{-1}} = L_{AA^{-1}} = L_{I_n} = I_{\mathbb{R}^n}$

as well as $L_{A^{-1}} \circ L_A = L_{A^{-1}A} = L_{I_n} = I_{\mathbb{R}^n}$.
Thus $L_{A^{-1}}$ is the inverse to L_A and so L_A is invertible. Conversely, assume ②. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the inverse of L_A. Set $\beta = \{\text{std basis for } \mathbb{R}^n\}$ and $B = [T]_\beta$; then $T = L_B$ since $[L_B]_\beta = B = [T]_\beta$. So

$L_{AB} = L_A \circ L_B = L_A \circ T = I_{\mathbb{R}^n}$, which gives $AB = I_n$. Similarly $L_{BA} = L_B \circ L_A = T \circ L_A = I_{\mathbb{R}^n} \Rightarrow BA = I_n$. So A is indeed invertible with inverse B. So we've shown ① $\Leftrightarrow$ ②.

Next we show ① $\Rightarrow$ ③. If A is invertible and $Ax = 0$, then

$$A^{-1}(Ax) = A^{-1}0 = 0$$

As $A^{-1}(Ax) = (A^{-1}A)x = I_n x = x$,

this gives $x = 0$. Thus $\mathcal{N}(A) = \{0\}$.

Finally, assume 3. As

$$\mathcal{N}(L_A) = \{x \in \mathbb{R}^n \mid \underset{\overset{\parallel}{Ax}}{L_A(x)} = 0\} = \mathcal{N}(A)$$

we get $\mathcal{N}(L_A) = 0$ which forces L_A to be 1-1 and hence onto by Theorems 2.4 and 2.5 of our text (which you used on the last HW). Thus L_A is invertible and so 3 $\Rightarrow$ 2 $\Rightarrow$ 1 ▨

<u>How do we compute A^{-1}?</u> Set $A^{-1} = B$

$= \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$. Since $AB = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix} = I_n$,

we get $Ab_i = e_i$ for each i.

Thus, we can find b_i by solving the linear system $\mathcal{LS}(A, e_i)$. To solve all n systems at once, use this trick:

<u>Ex</u>: Find inverse of $A = \begin{pmatrix} 4 & 16 & 5 \\ 6 & 25 & 8 \\ 1 & 3 & 1 \end{pmatrix}$.

Consider the "super augmented" matrix

$$\left(\begin{array}{ccc|ccc} 4 & 16 & 5 & 1 & 0 & 0 \\ 6 & 25 & 8 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now do row ops to put this in reduced row echelon form. One gets

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -7 & 4 & 4 \end{array}\right)$$

b_1 b_2 b_3

and so $A^{-1} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & -2 \\ -7 & 4 & 4 \end{pmatrix}$.

<u>Note</u>: If A is <u>not</u> invertible, will end up with left-hand part $\neq I_n$ and

one of the systems being inconsistent.

Application: When know A^{-1} can use it to solve any $\mathcal{LS}(A, b)$ since $Ax = b \Rightarrow x = A^{-1}b$.

Suppose $A \in M_{m\times n}(\mathbb{R})$. We can associate 3 subspaces to A: [The 1st two we've seen before.]

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$RowSp(A) = \mathrm{span}(\text{rows of } A) \subseteq \mathbb{R}^n$$

$$ColSp(A) = \mathrm{span}(\text{cols of } A) \subseteq \mathbb{R}^m$$

In terms of $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \longmapsto Ax$ have

$$\mathcal{N}(A) = \mathcal{N}(L_A) \text{ and } ColSp(A) = \mathcal{R}(L_A)$$

where the latter is because $\mathcal{R}(L_A)$ is spanned by the the image of any basis.

By the Dimension Thm applied to L_A we have

$$\dim \mathcal{N}(A) + \dim ColSp(A) = \dim \mathbb{R}^n = n.$$

Now, back in Lecture 12, we saw

$$\dim \mathcal{N}(A) + \dim RowSp(A) = n$$

by using that for matrices B in reduced row echelon form one has

$$\dim \mathcal{N}(B) = \#\text{ of non-pivot cols}$$

$$\dim RowSp(B) = \#\text{ of nonzero rows} = \#\text{ of pivot cols}.$$

[Recall that row equivalent matrices have the same null space and row space.]

Thus we get

<u>Cor</u>: $\dim RowSp(A) = \dim ColSp(A)$.

This number is usually called the <u>rank</u> of A.