Lecture 25: Diagonalization and eigenstuff.

[§5.1 of FIS]

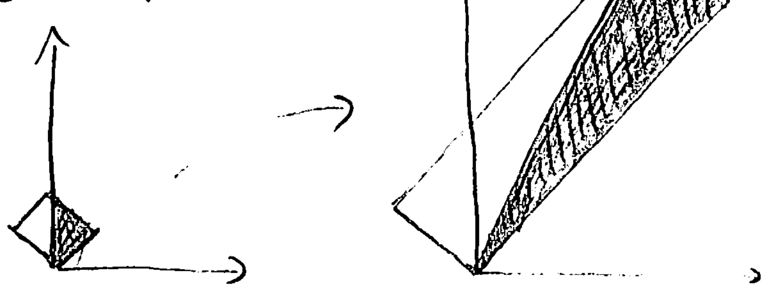
Diagonal matrix: $A \in M_{n \times n}$ with $A_{ij} = 0$ when $i \neq j$

Ex: $\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$ Non Ex: $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

Linear operator: A linear transformation $T: V \longrightarrow V$.

Key Def: A linear operator T of a vector space V is diagonalizable if there is basis β for V with $[T]_\beta$ diagonal.

Ex: Set $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ and $T = L_A$. In Lecture 18 saw that if $\beta' = \{v_1 = (1,1), v_2 = (-1,1)\}$ then $[T]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$.



Non Ex: $B = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ and $S = L_B$.

Turns out, best you can do is

$$[S]_\gamma = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ where } \gamma = \{(1,1),\ (1,2)\}$$

[How do we tell the difference? How do we find the right basis?]

If $\beta = \{v_1, \dots, v_n\}$ is a basis for V where

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \text{ then } T(v_i) = \lambda_i v_i$$

for each i.

Def: An eigenvector for a linear op T is a $v \neq 0$ in V where there is a scalar λ with $T(v) = \lambda v$. The scalar λ is called the eigenvalue assocatiated to v.

[Requirement that $v \neq 0$ is so that 0 is not a eigenvector for all λ.]

<u>Ex</u>: $T = L_A$ with $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$. Then

a) $v = (1,-1)$ is a eigenvector with eigenvalue 2 as $T(1,-1) = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) $w = (1,2)$ is not an eigenvector as

$$T(1,2) = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

<u>Thm</u>: A linear operator T of V is diagonalizable if and only if there is a basis of V consisting of eigenvectors of V.

<u>Pf</u>: ($\Rightarrow$) Clear from above.

($\Leftarrow$) Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis of V where $T(v_i) = \lambda_i v_i$ for some scalar λ_i.

Then

$$[T]_\beta = \begin{pmatrix} | & \\ [T(v_1)]_\beta & \cdots \\ | & \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Def: A matrix A is diagonalizable when L_A is diagonalizable. An eigenvector for A is one for L_A.

Alternatively, an eigenvector $v \in \mathbb{R}^n$ for $A \in M_{n \times n}$ is one where $Av = \lambda v$ for some $\lambda \in \mathbb{R}$. Also A is diagonalizable $\iff$ there exists an invertible $Q \in M_{n \times n}$ with $Q^{-1} A Q = (\text{diagonal})$.

Thm: Suppose $A \in M_{n \times n}(\mathbb{R})$. Then $\lambda \in \mathbb{R}$ is an eigenvalue for A if and only if $\det(A - \lambda I_n) = 0$.

Ex: $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ $\quad A - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \xrightarrow{\det} 0$

$$A - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \xrightarrow{\det} -5$$

[Before proving, we need a lemma.]

Lemma: $v \in \mathbb{R}^n$ is an eigenvector of $A \in M_{n\times n}(\mathbb{R})$ with eigenvalue $\lambda \iff v \in \mathcal{N}(A - \lambda I_n)$ and $v \neq 0$.

Proof:
$$Av = \lambda v \iff Av - \lambda v = 0$$
$$\iff Av - \lambda(I_n v) = 0$$
$$\iff (A - \lambda I_n)v = 0 \qquad \blacksquare$$

Lemma: $B \in M_{n\times n}(\mathbb{R})$. Then $\mathcal{N}(B) \neq \{0\}$ if and only if $\det(B) = 0$.

Proof: If $\det(B) \neq 0$ then B is invertible, and so $Bv = 0 \Rightarrow v = B^{-1}0 = 0$ and so $\mathcal{N}(B) = \{0\}$. If instead $\det(B) = 0$, then B is not invertible, and so $\operatorname{rank}(B) < n$ and $\operatorname{nullity}(B) > 0$. In particular $\mathcal{N}(B) \neq \{0\}$. $\blacksquare$

<u>Proof of theorem</u>: By the 1^{st} lemma,

λ is an eigenvalue for $A \iff N(A - \lambda I_n) \neq \{0\}$.

$\iff \det(A - \lambda I_n) = 0$. ∎

2nd Lemma

<u>Def</u>: The <u>characteristic polynomial</u> of $A \in M_{n \times n}$

is $f(t) = \det(A - tI_n)$.

<u>Ex</u>: $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ $\quad A - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 5-t & 3 \\ 3 & 5-t \end{pmatrix}$

Taking det gives

$$\begin{aligned} f(t) &= (5-t)(5-t) - 9 \\ &= t^2 - 10t + 16 = (t-2)(t-8) \end{aligned}$$

<u>Thm</u>: The eigenvalues of A are exactly the roots of its characteristic polynomial.