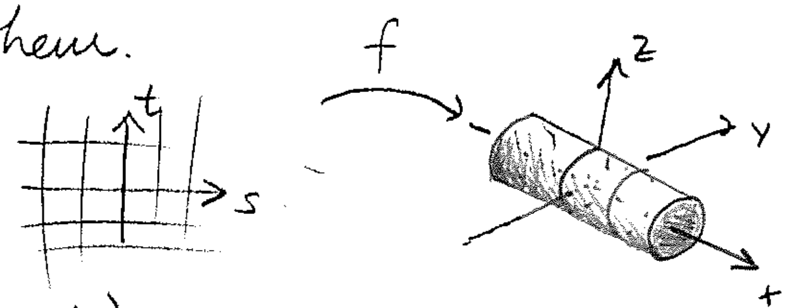
Lecture 12: Linear Transformations, an introduction [FIS Ch 2]

Now that we've studied vector spaces, we look at functions between them.

Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

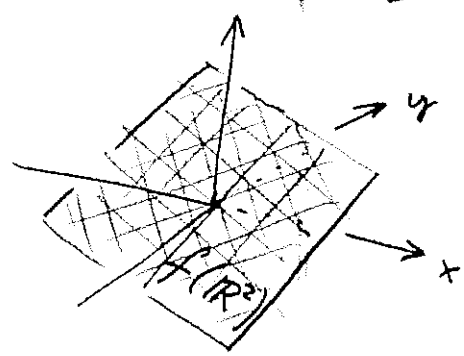
$$f(s,t) = (s, \cos t, \sin t)$$



[In this class, we'll focus on the simplest kind of functions between vector spaces, for example.]

Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(s,t) = (s, t, s+t)$$



Def: Suppose V and W are vector spaces over $\mathbb{R}$. A function $T: V \longrightarrow W$ is a *linear transformation* if for all $v_1, v_2 \in V$ and $a \in \mathbb{R}$ we have

a) $T(v_1 + v_2) = T(v_1) + T(v_2)$.

b) $T(av_1) = aT(v_1)$.

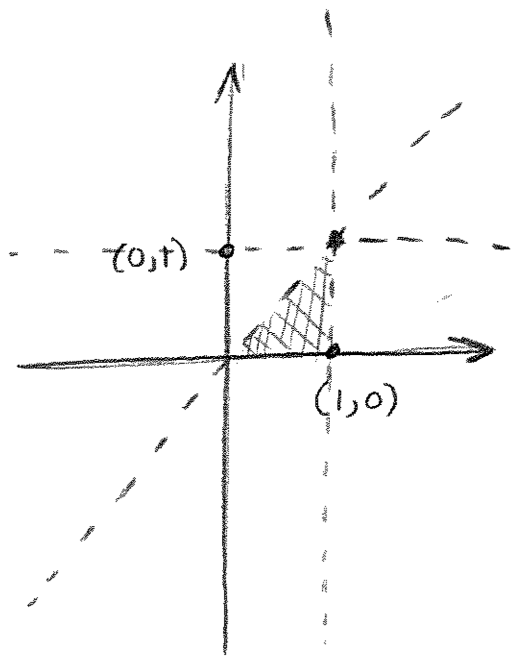
Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

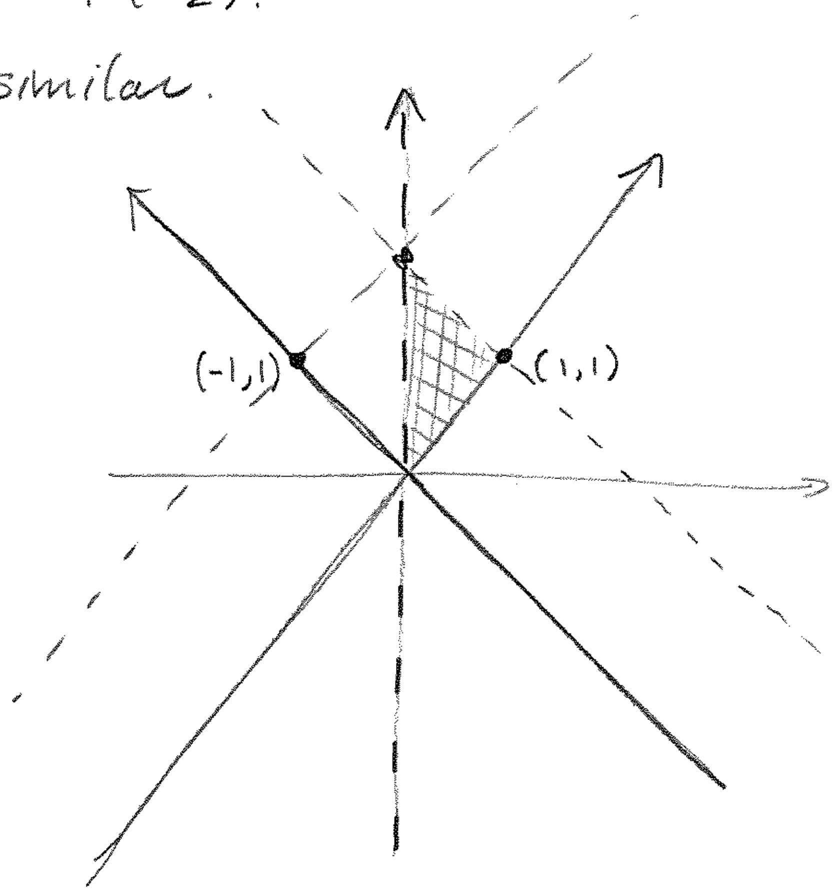
$T(x,y) = (x-y, x+y)$ Let's check this is a linear trans.

$v_1 = (x_1, y_1)$ $v_2 = (x_2, y_2)$

$$
\begin{aligned}
T(v_1 + v_2) &= T((x_1+x_2, y_1+y_2)) \\
&= ((x_1+x_2)-(y_1+y_2), (x_1+x_2)+(y_1+y_2)) \\
&= (x_1-y_1, x_1+y_1) + (x_2-y_2, x_2+y_2) \\
&= T(v_1) + T(v_2).
\end{aligned}
$$

Scalar mult is similar.





$T(x,0) = (x,x)$

$T(0,y) = (-y,y)$

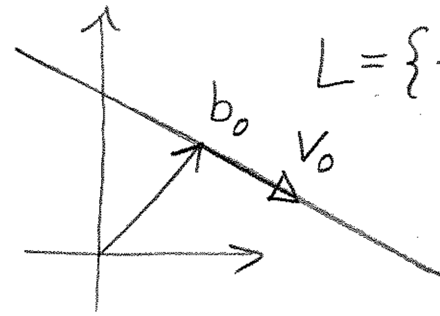
$T(t,t) = (0,2t)$

$T(1,1) = 2$

By prop 6, namely $T(av) = aT(v)$ see that T must send a line through 0 to another one. In fact, this is true for

<u>any</u> line in $\mathbb{R}^2$:

$L = \{tv_0 + b_0 \mid t \in \mathbb{R}\}$



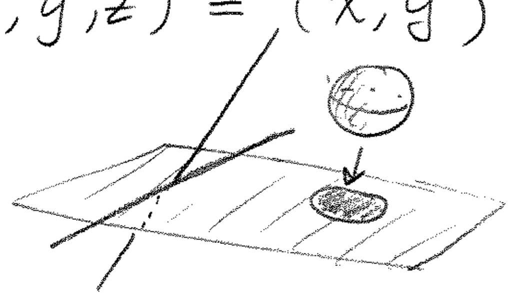
$T(tv_0 + b_0) =$

$$T(tv_0) + T(b_0) = t\,T(v_0) + T(b_0)$$

[Draw some examples]

<u>Ex</u>: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad T(x,y) = (x, y, x+y)$ [From before]

<u>Ex</u>: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T(x,y,z) = (x,y)$



<u>Ex</u>: $T: P(\mathbb{R}) \rightarrow P(\mathbb{R})$

$$T(f(x)) = f'(x)$$

$$T(x^3 + 2x^2 + 5x + 1) = 3x^2 + 4x + 5$$

[The fact that this is a linear transformation is one of the first things you learned about the derivative in Calc I]

<u>Ex</u>: $T: M_{m\times n}(\mathbb{R}) \longrightarrow M_{n\times m}(\mathbb{R})$

$$A \underset{\text{"sends"}}{\longmapsto} A^t$$

[Used implicitly to show symmetric matrices are a subspace of $M_{n\times n}(\mathbb{R})$.]

<u>Def</u>: Suppose $T: V \to W$ is a linear transformation.

The nullspace (kernel) of T is

$$\mathcal{N}(T) = \{ v \in V \mid T(v) = 0_W \}.$$

The range (image) of T is

$$R(T) = \{ T(v) \mid v \in V \}$$

<u>Thm</u>. These are subspaces of V and W respectively.

<u>Pf</u>: For $\mathcal{N}(T)$:

i) $T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$,

so $0_V \in \mathcal{N}(T)$.

ii) Suppose v_1, v_2 are in $\mathcal{N}(T)$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0_W + 0_W = 0_W$$

and so $v_1 + v_2 \in \mathcal{N}(T)$.

iii) Suppose $a \in \mathbb{R}$ and $v_1 \in \mathcal{N}(T)$. Then

$$T(av_1) = aT(v_1) = a \cdot 0_W = 0_W$$

and so $a v_1 \in \mathcal{N}(T)$.

The case of $R(T)$ is similar. $\blacksquare$

[Go back to examples, figure out nullsp, range.]

<u>Dimension Theorem</u>: Suppose $T: V \to W$ is a linear transformation. If V is finite dim'l, then

$$\underbrace{\dim(\mathcal{N}(T))}_{\text{nullity}} + \underbrace{\dim(R(T))}_{\text{rank}} = \dim V.$$

[Again revisit the examples.]

[If time remains, mention how linear trans. can be encoded in matrices, and hint at the connection $\mathcal{N}(T) \longleftrightarrow \mathcal{N}(A)$ + $R(T) \longleftrightarrow R(A)$]