

Lecture 5: Using matrices to solve linear systems (1)

Reference: [SRREF of Breezer]

Last time: System of linear equations:

Variables: x_1, x_2, \dots, x_n

Equations:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are in \mathbb{R} .

To a linear system, we associate a coefficient matrix $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$,

a vector of constants $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ ← column vector.

and an augmented matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

$$\underline{\text{Ex:}} \quad x_1 + 2x_2 + 2x_3 = 4$$

$$x_1 + 3x_2 + 3x_3 = 5$$

$$2x_1 + 6x_2 + 5x_3 = 6$$

(2)

This system has $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

The augmented matrix $(A|b)$ is $\begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{pmatrix}$.

Decoding:

Given an $m \times n$ matrix A and an m entry column vector b , we associate a linear system $\mathcal{L}(A, b)$. Also works for augmented matrices.

$$\underline{\text{Ex:}} \quad \mathcal{L}\left(\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}\right) = \begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + x_2 = 0 \end{cases}$$

Row operations: These take a matrix M and give a new matrix of the same size.

- ① Swap two rows of M
- ② Multiply all entries in a row by some nonzero c in \mathbb{R}
- ③ Add a multiple of one row to another.

[Hint at use for solving lin. systems.]

Ex: $M = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 1 & 3 & 3 & 5 \end{pmatrix}$

$\xrightarrow{-3R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ -3 & -9 & -9 & -15 \end{pmatrix} \xrightarrow{3R_1 + R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & -3 & -3 & -3 \end{pmatrix}$

$\xrightarrow{-\frac{1}{3}R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Def: A matrix M is row equivalent to N if there is a sequence of row operations that turns M into N .

Note: On HW you'll show this is symmetric in M and N .

Thm: If M is row equivalent to N , then the linear systems $LS(M)$ and $LS(N)$ have the same solution sets.

Pf: Just need to check that a single row op does not change the solution set of the corresp. linear system. Row op ① is fine.

[since it corresponds to reordering two equations]

Row op ② just scales the assoc. eqn. For row op ③, it is enough to consider the case of adding rows without any scaling.

[The general case follows by using row op ②]

On the equation side, have

$$\begin{matrix} \textcircled{E_1} & a_1x_1 + \dots + a_nx_n = b \\ & a'_1x_1 + \dots + a'_nx_n = b' \end{matrix} \rightsquigarrow$$

$$\begin{matrix} \textcircled{E_2} & a_1x_1 + \dots + a_nx_n = b \\ & (a'_1+a_1)x_1 + \dots + (a'_n+a_n)x_n = b+b' \end{matrix}$$

If numbers (x_1, \dots, x_n) satisfy (E_1) (5)

then they also sat (E_2) . Conversely if

they sat (E_2) they also sat (E_1) since we can subtract the first eqn from the second equation of (E_2) .

So all three kinds of row ops don't change the sol set of the corresp linear sys. \square

Ex. (Continuation)

$$M \xrightarrow[\text{ops}]{\text{row}} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_2} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_2 + R_1} \begin{pmatrix} 1 & 0 & 1 & 6 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 & 6 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[\begin{matrix} R_2 + R_1 \\ R_2 + R_3 \end{matrix}]{R_2 + R_1} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & -3 \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{\text{ZS}} \begin{matrix} x_1 = 2 \\ x_3 = 4 \\ x_2 = -3 \end{matrix} \quad \text{So } (x_1, x_2, x_3) = (2, -3, 4)$$

is the unique soln to our original system: ⑥

$$x_1 + 2x_2 + 2x_3 = 4$$

$$x_1 + 3x_2 + 3x_3 = 5$$

$$2x_1 + 6x_2 + 5x_3 = 6.$$

as you can (and should!) double check:

Goal: Develop systematic procedure for solving such systems, by finding matrices like these:

Reduced Row-Echelon Form: A matrix where:

- ① All zero rows are at the bottom.
- ② The leftmost entry of every row is 1, and is called a leading 1.
- ③ A leading 1 is the only nonzero entry in its column.
- ④ Suppose entries (i, j) and (s, t) are leading 1's. If $s > i$ then $t > j$.