Lecture 20: Intro to determinants.

[§4.1 and §4.2 of FIS]

[Next few lectures will discuss a key tool for working with square matrices.]

Determinant: $\det : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$.

Fun facts: ① $\det(A) \neq 0 \iff A$ is invertible.

② $\det(AB) = \det(A)\det(B)$.

③ $\det(A)$ tells us how $L_A : \mathbb{R}^n \to \mathbb{R}^n$ changes volumes of objects.

④ det is not linear (for $n > 1$) but "n-multilinear."

[In vector calculus, saw det's of 2×2 and 3×3 matrixes, e.g. in cross products and when changing coordinates in multivar integrals. Let's start at the beginning...]

$n=1$: $\det : M_{1\times 1}(\mathbb{R}) \longrightarrow \mathbb{R}$
$(a) \longmapsto a$

[Discuss props above...]

(3) $L_{(a)}: \mathbb{R} \longrightarrow \mathbb{R}$
$(x) \longmapsto (ax)$

<u>$n=2$:</u> For $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ set $\det(A) = ad - bc$.

$$\det\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} = 2 \quad \det\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} = -2 \quad \det\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix} = -1$$

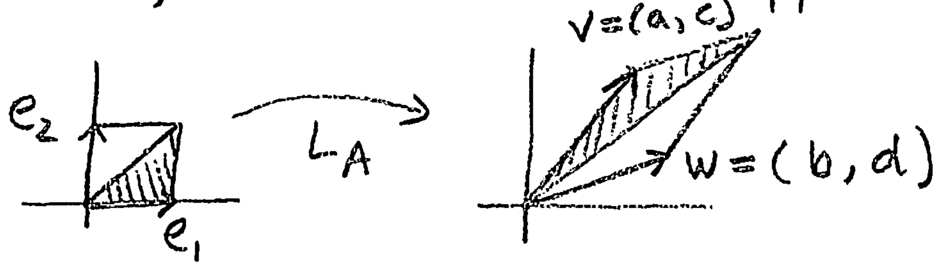
(4) If A, B, C are the above matrices, then
$C = A + B$ but $\det(C) = -1 \neq 0 = \det(A) + \det(B)$.

(2) Homework.

(1) ($\Rightarrow$) If $\det(A) \neq 0$, you will check on the HW
that $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

($\Leftarrow$) If A is invertible, then $\det(A \cdot A^{-1})$
$= \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ and by (2) we have
$\det(A \cdot A^{-1}) = \det(A)\det(A^{-1})$, so
$\det(A) \neq 0$ and moreover $\det(A^{-1}) = (\det A)^{-1}$.

3 First, calculate what happens to areas here:



[Query: how would you do this in Calc III?]

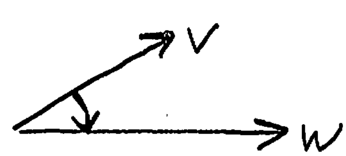
<u>Thm</u>: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $v=(a,c)$, $w=(b,d)$.

Then $|\det(A)|$ = area of parallelogram spanned by v and w.

Moreover, the sign of $\det(A)$ depends on whether (v,w) is a positively or negatively oriented basis for $\mathbb{R}^2$.

<u>positive</u>: counter-clockwise

<u>negative</u>: clockwise



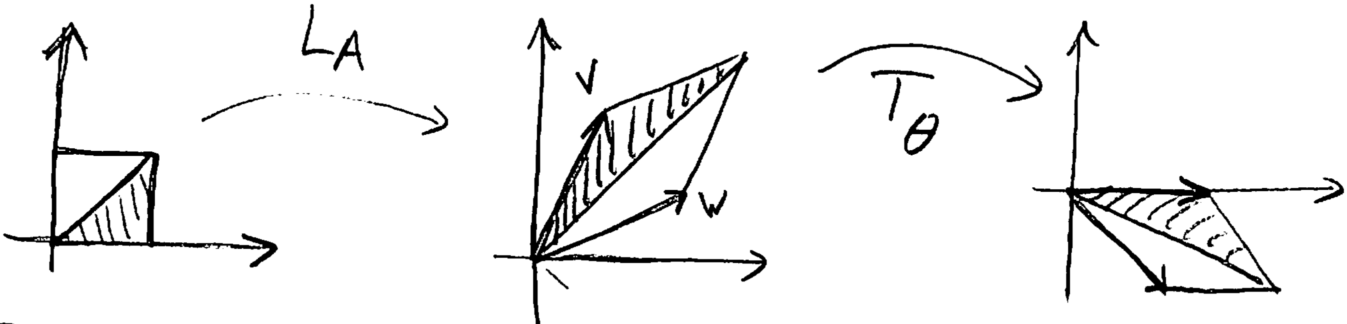
Proof: [Change of coordinates to the rescue...]

From HW, $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has $[T_\theta]_{std} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

rotate by θ counterclockwise

Note that $\det([T_\theta]_{std}) = \cos^2\theta + \sin^2\theta = 1$.

Choose θ so that $T_\theta(v)$ is on the positive x-axis, and set $U = [T_\theta]_{std}$. Now



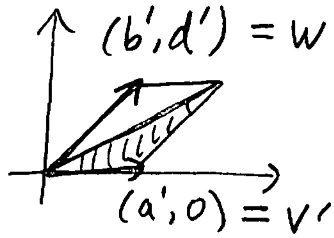
Now being a rigid rotation T_θ doesn't change areas or the orientation of two vectors. As

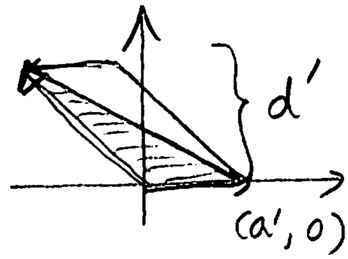
$$\det([T_\theta \circ L_A]_{std}) = \det([T_\theta]_{std}) \det([L_A]_{std})$$
$$= \det(A)$$

it suffices to prove the theorem for

$$A' = [T_\theta \circ L_A]_{Std} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}.$$

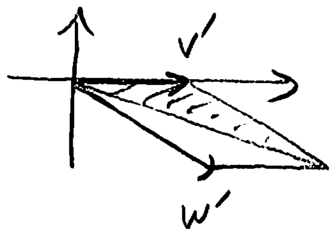
<u>Case $d' > 0$:</u>

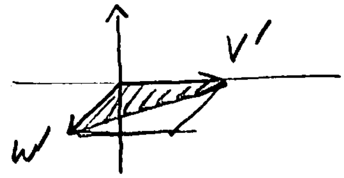




Then (v', w') is positively oriented, and the area is $a' \cdot d' = \det(A')$.

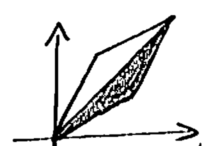
<u>Case $d' < 0$:</u>





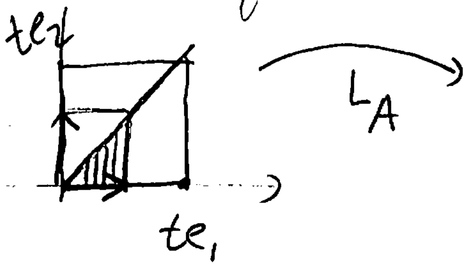
Then (v', w') is negatively oriented and the area is $a' \cdot (-d') = -\det(A')$. $\blacksquare$

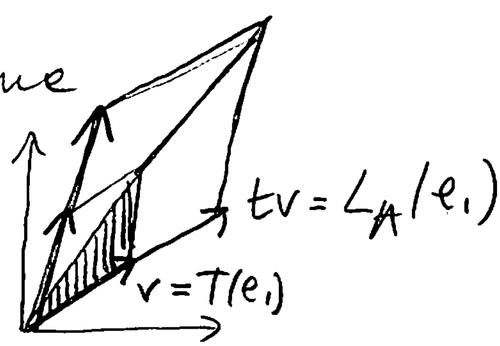
So now we know how $L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ changes the area of



$$1 \longmapsto |\det(A)|$$

Now consider a square of some other size t

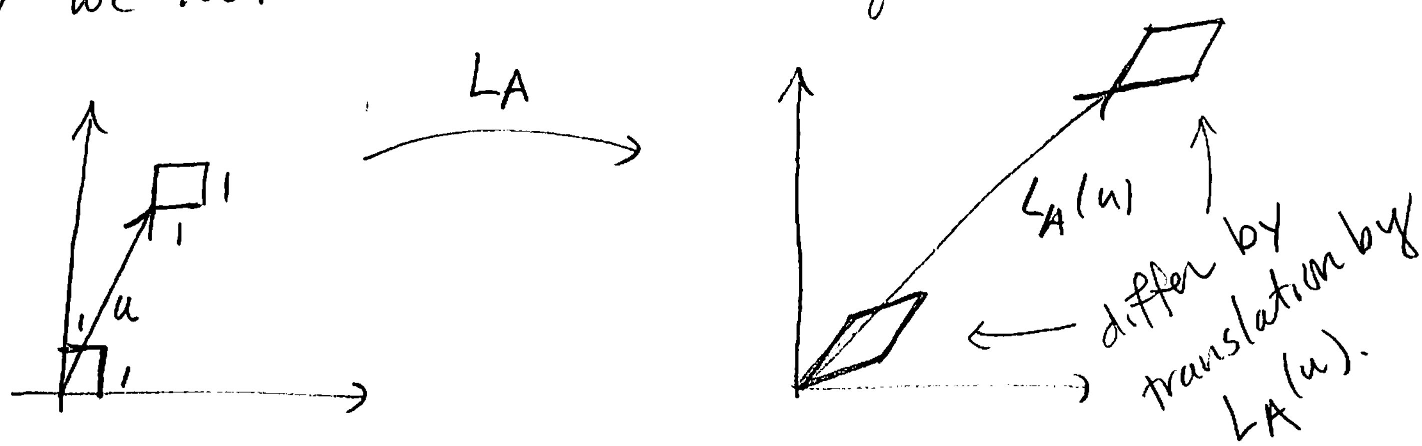




Because of linearity, have

$$L_A\left(\underset{t}{\square}\,t\right) = t \cdot L_A\left(\underset{1}{\square}\,1\right)$$

which has area $= t^2 \,\mathrm{Area}\left(L_A\left(\underset{1}{\square}\,1\right)\right)$. So the ratio of areas is also $|\det(A)|$. Similarly, if we look at a translated square



linearity of L_A shows that its area changes by the same ratio.

Next time: Bigger matrices!