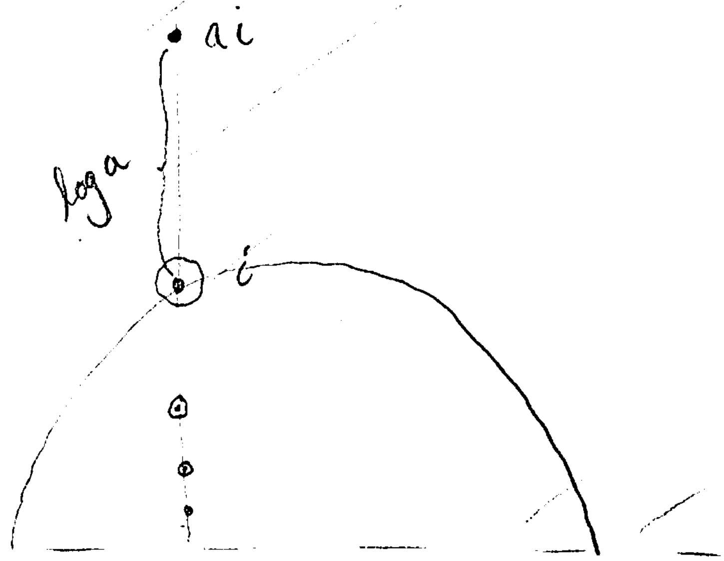
Lecture 2: Basic examples.

(Throughout think $\mathbb{E}^2/\mathbb{Z}^2$ = )

2D: $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

$g_{\mathbb{H}^2} = \frac{1}{y^2} g_{\mathbb{E}^2}$



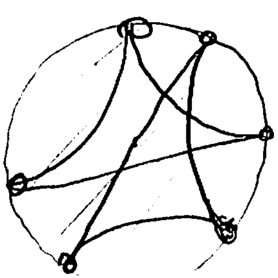
$\operatorname{Isom}^+ \mathbb{H}^2 = \text{Möb}(\mathbb{H}^2)$

$= \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{array} \right\}$

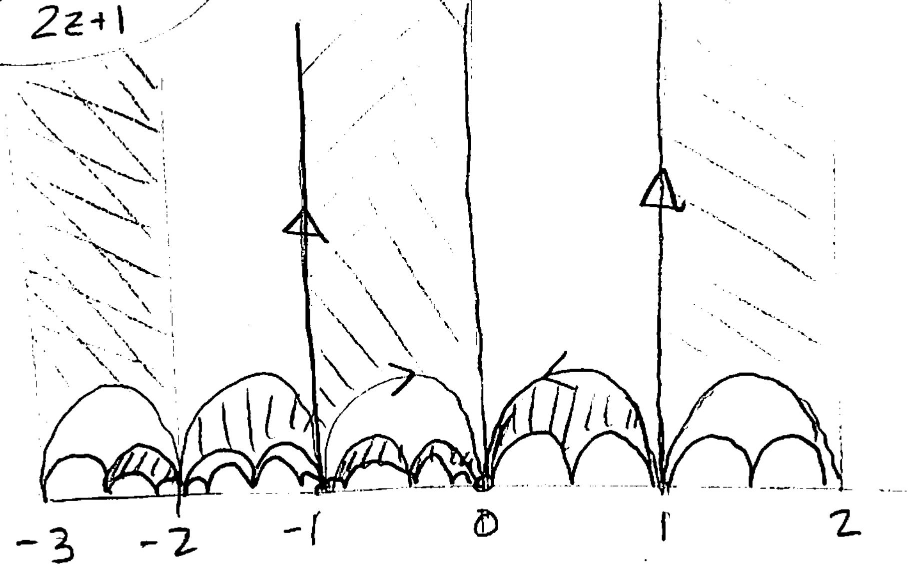
$= SL_2\mathbb{R} / \{\pm I\} = PSL_2\mathbb{R}$

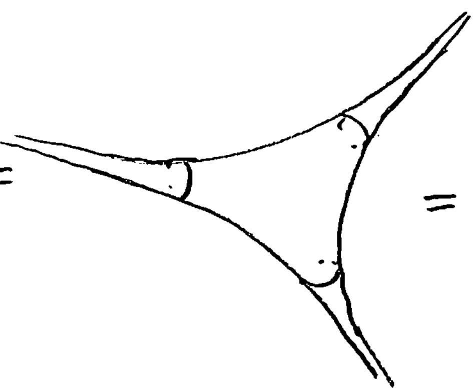
$$\Lambda = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle = \ker\left(PSL_2\mathbb{Z} \to PSL_2\mathbb{F}_2\right)$$

$z \mapsto z+2 \qquad z \to \frac{z}{2z+1}$

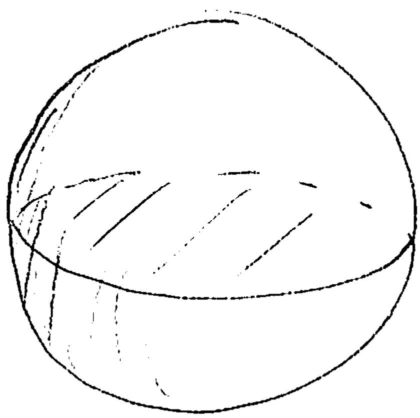


How many ideal Δ's are there?



$\mathbb{H}^2 / \Lambda =$  $= S^2 \setminus (3\text{pts})$

Not cpt, but area is 2π.



$\mathbb{H}^3 = \{ |x| < 1 \}$

$\mathrm{Isom}^+(\mathbb{H}^3) = \text{Möb}(S^2_\infty) = PSL_2\mathbb{C}$

[Mention hyperboloid model]

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2-i & 1 \\ 2i & i \end{pmatrix}, \begin{pmatrix} 3 & 2i \\ 2i & -1 \end{pmatrix} \right\rangle$$

$M = {}_{\Gamma}\backslash \mathbb{H}^3 = S^3 \setminus$  = Borromean rings

$\mathrm{Vol} \approx 7.327724753\ldots$

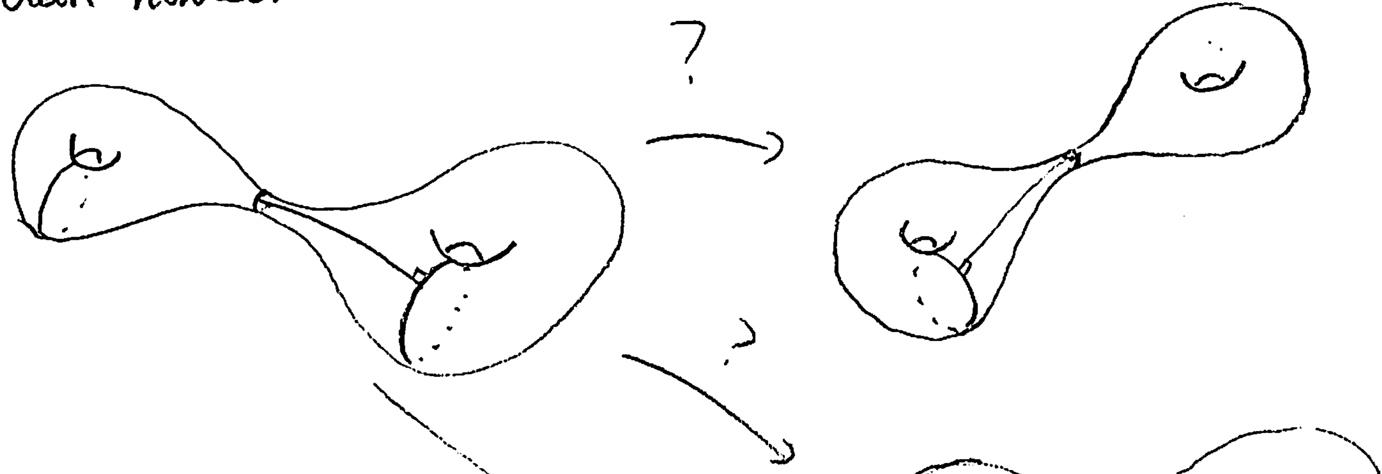
Mostow Rigidity: Suppose M and N are finite-volume hyperbolic n-mflds and $n \geq 3$. If $\pi_1 M \cong \pi_1 N$ then M and N are isometric.

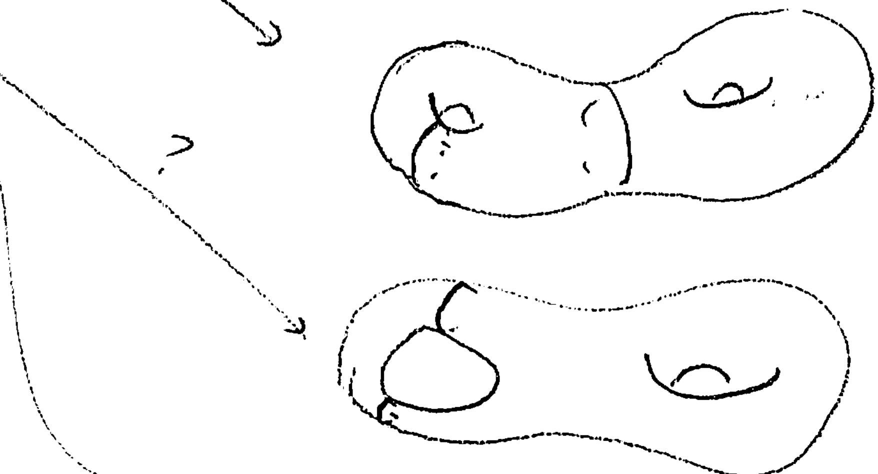
"Topology = Geometry in dimension 3" – W. Thurston.

Connect to homeomorphism problem: Input M, N clsd 3-mflds

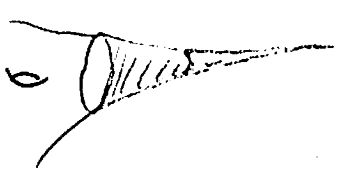
1. Find their topological and geometric decomp.
2. Non-hyp pieces are classified.
3. For hyperbolic pieces, need to test for isometry.

Key point: Much easier to test for isometry than homeom:





Cusps: If S is a hyp. surface of finite area, the ends are all cusps



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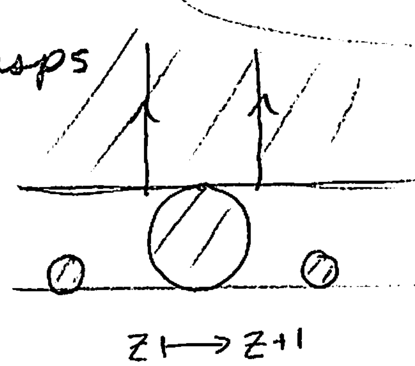
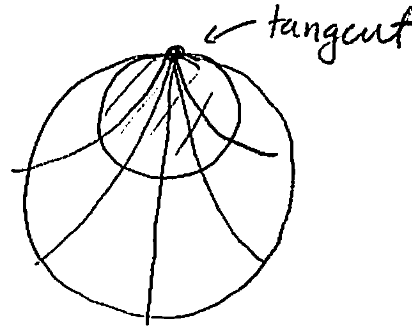
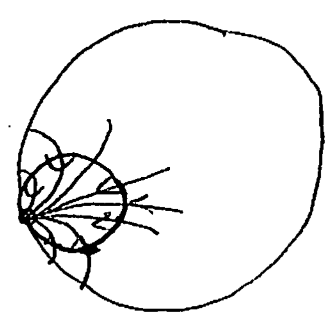


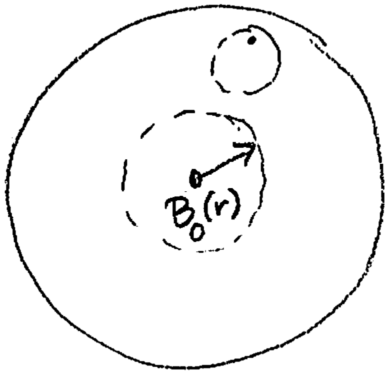


If $\mathbb{H}^2$, region $\{\operatorname{im}(z) \geq y_0\}$ is a horodisc and $\{\operatorname{im}(z) = y_0\}$ is a horocircle.

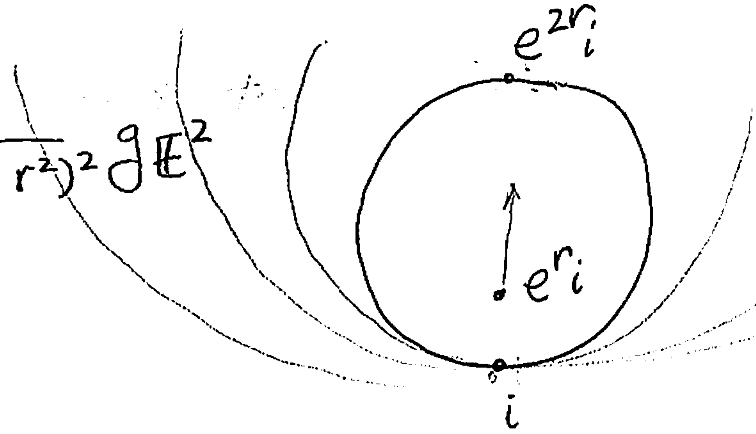
Same for any image under an isometry.

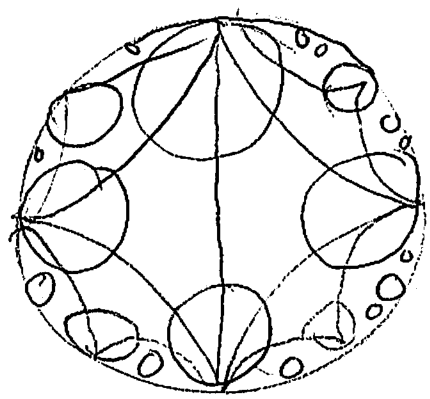


Also horodiscs are limits of closed metric balls.



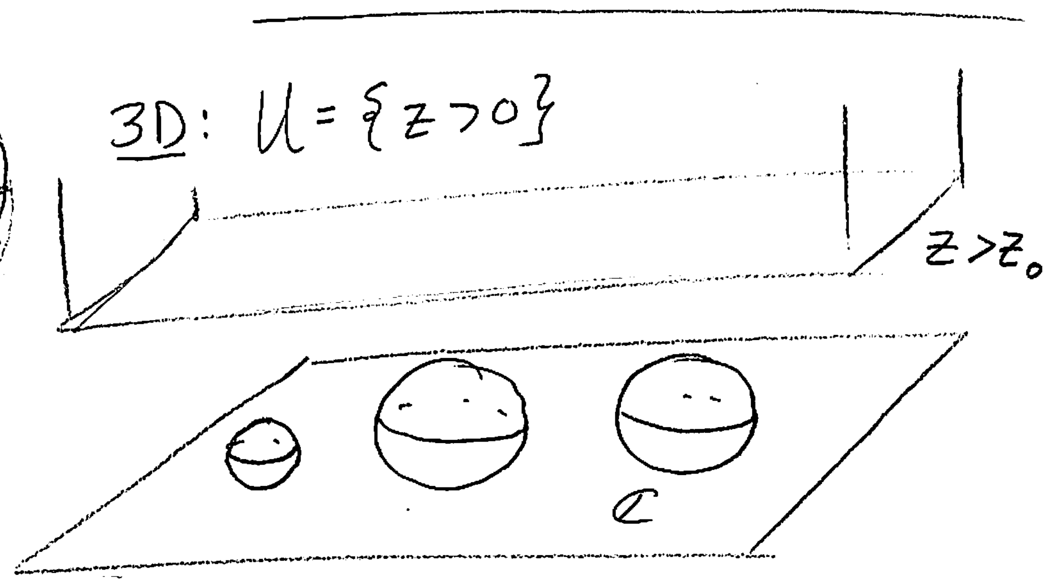
$\frac{1}{(1-r^2)^2} g_{\mathbb{E}^2}$



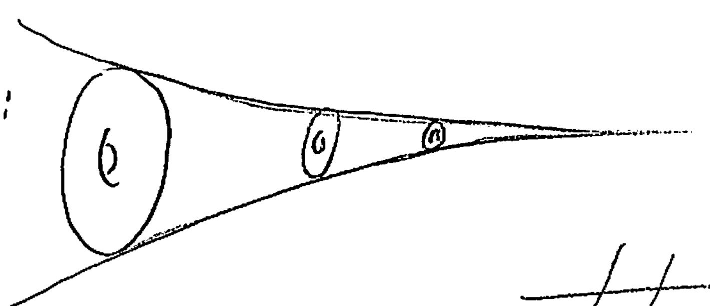


$\frac{1}{z^2} g_{\mathbb{E}^3}$

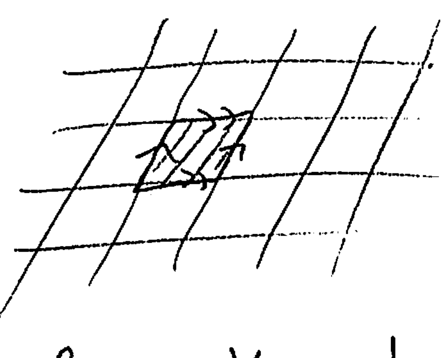
3D: $\mathcal{U} = \{z > 0\}$



Cusp:



Λ = lattice in $\mathbb{E}^2$



gives isom of $\mathbb{H}^3$ fixing the pt at ∞.

Result is a chimney...

On to SnapPy...