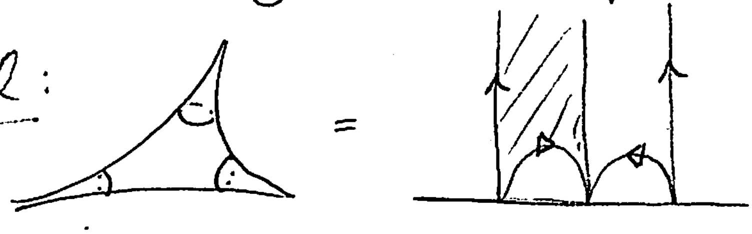
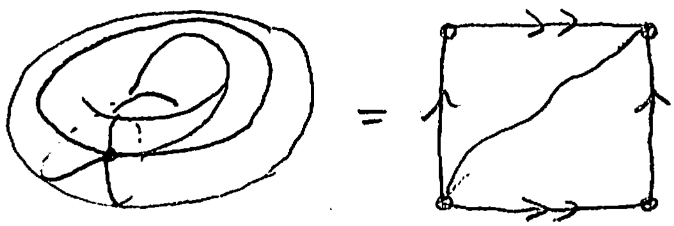
Lecture 4: Triangulations to hyperbolic structures

Recall:

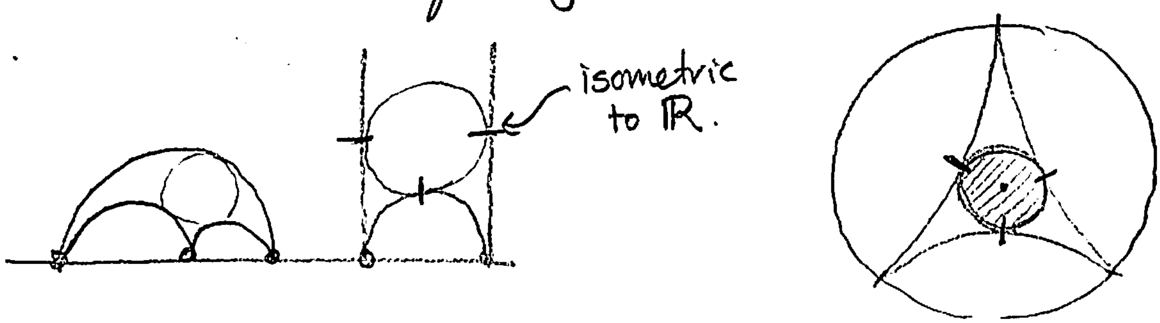


Ideal triangulation: Built from triangles with sides identified so that vertices = punctures.

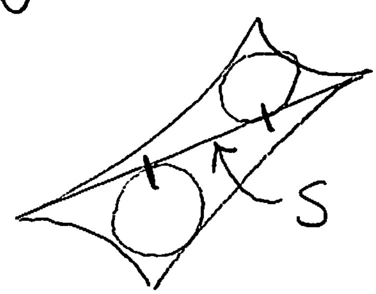
Ex:



In $\mathbb{H}^2$, there is a unique geodesic ideal tri, up to isom.



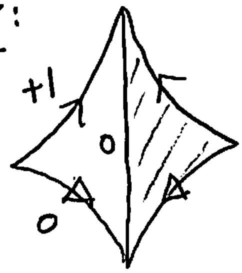
To put a hyp. str on a ideal triang, need to chose a shear for each edge



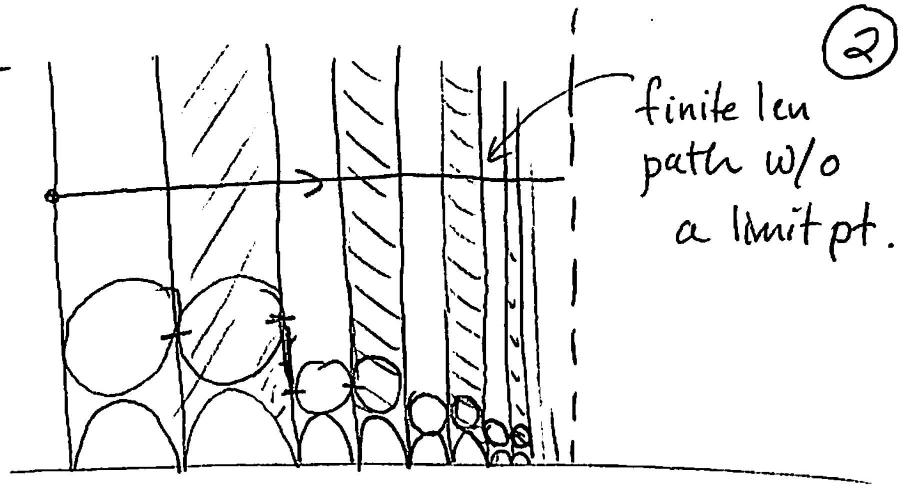
Any choice gives a hyp. metric on the surface.

Warning: It may not be complete!

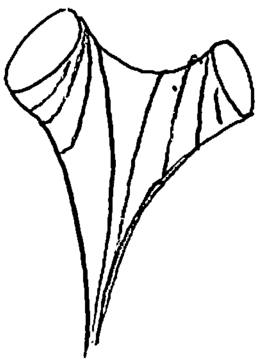
Ex:



In $\mathbb{H}^2$:



Secretly: Metric completion is a hyp surface with geod. bdry and one cusp.



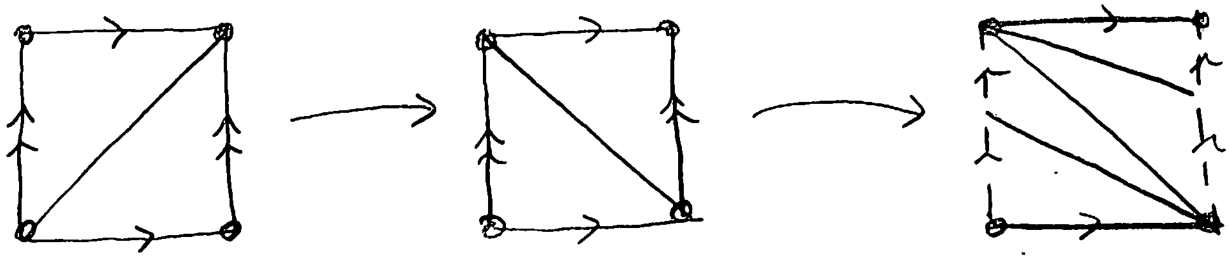
Back to 3D: N^3 cpt with ∂N a union of tori, $M = \text{int}(N)$.

An ideal triang. $\mathcal{T}$ of M is a cell complex built from finitely many tet with faces glued in pairs where $\mathcal{T} \setminus \mathcal{T}^{(0)} \cong M$ ($\mathcal{T}$ = cpt of M with one pt per end).

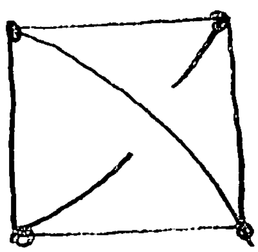
Ex: $S^3 \setminus$  has an ideal tri with two tets.

Ex: f a homeo of $\Sigma_{g,n}$, $M_f = \Sigma \times I \big/ (x,1) \sim (f(x), 0)$

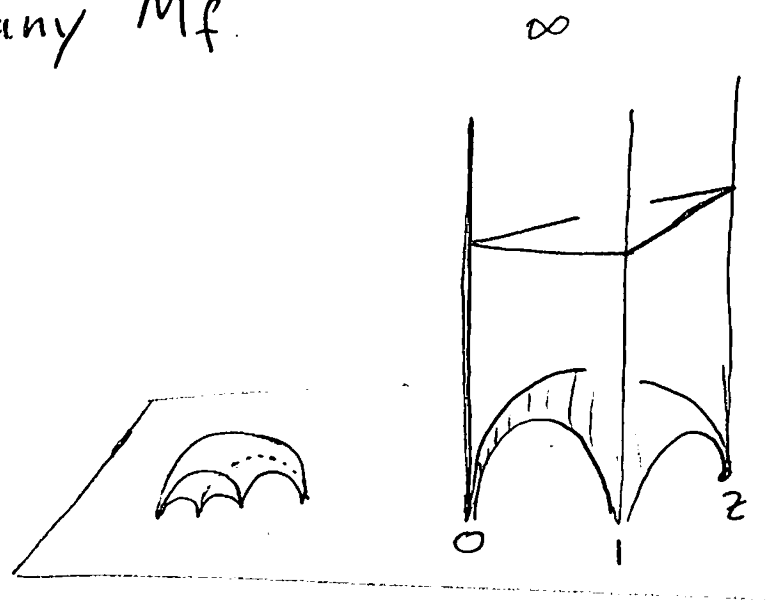
[M_f is hyp $\iff$ f is pseudo-Anosov.]



Can implement a flip by gluing in a tetrahedron



Can use this to give an ideal tri of any M_f.



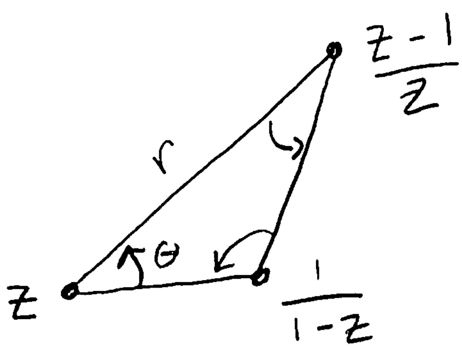
<u>Geometric ideal tets:</u>

moduli = cross ratio of 4 verts on $S^2_\infty = P^1(\mathbb{C})$.

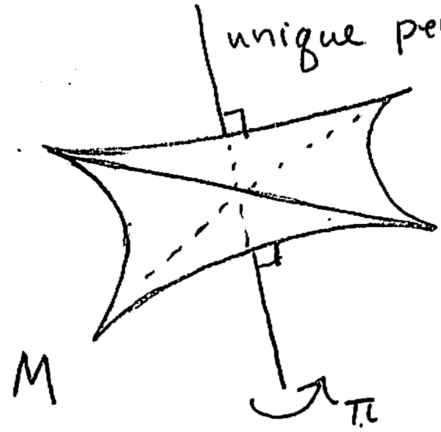
Call z the shape param. assoc. to the edge from 0 to ∞. [think "complex dihedral angle".]

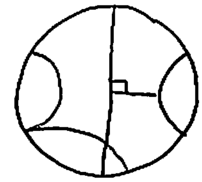
<u>Note</u>: The shape param. of any edge determines the rest.

$z = re^{i\theta}$.



Opposite edges have the same shape param:





Setting: $\mathcal{T}$ an ideal tri of M

z_i shape of tet i $\in \mathbb{C} \setminus \{0, 1, \infty\}$

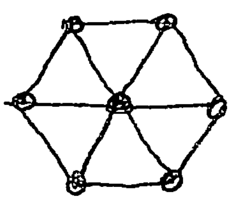
Q: When do these give a comp hyp str to M?

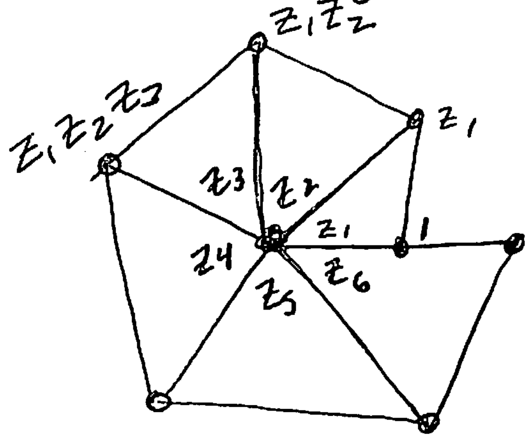
Note: Can always (uniquely) glue any pair of faces.

Edge eqn: [Motivation: $\sum$ dihedral angles $= 2\pi$]

Look down an edge:

Topological pic:





Geometric picture.

Need to require: $z_1 z_2 z_3 z_4 z_5 z_6 = 1$.

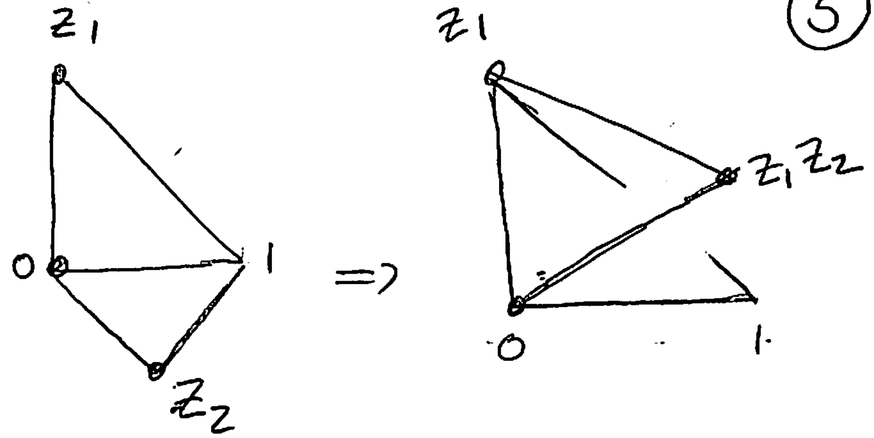
and also $\quad \mathrm{Im}(z_i) > 0$

and finally $\quad \sum \arg(z_i) = 2\pi$.

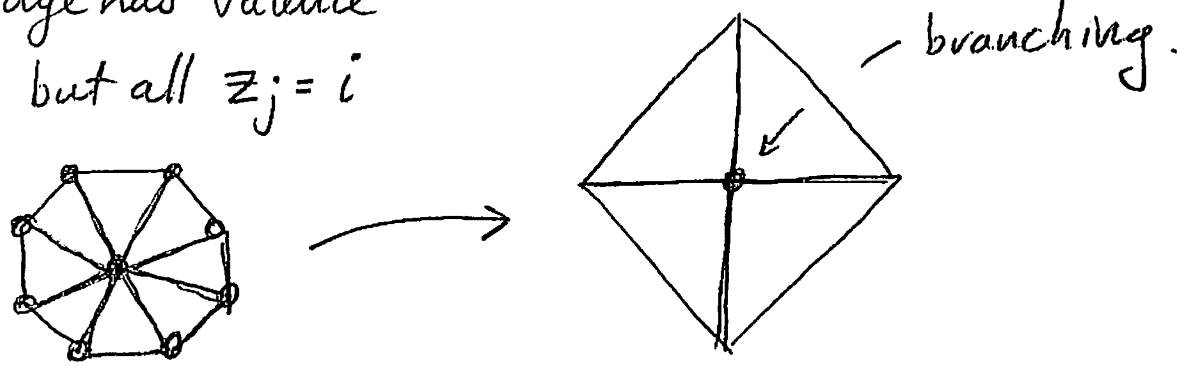
Reasons for last two:

a) $z_1 = i$

$z_2 = \overline{\zeta}_6 = \frac{1}{2}(1 - \sqrt{3}\,i)$



b) Edge has valence 8 but all $z_j = i$



Note: Any z_i sat all edge conditions gives a hyp. str. on M — may not be complete.