

Lecture 2: Vector spaces

①

[Go over syllabus.]

Ex: Vectors in \mathbb{R}^2 , \mathbb{R}^3 , or indeed \mathbb{R}^n .

Def: A vector space over \mathbb{R} is a set V with two operations

Addition: Assigns to each pair v, w in V a unique $v+w$ in V .

Scalar mult: Assigns to each a in \mathbb{R} and v in V a unique av in V .

where the following rules hold.

- 1) For all u, v in V , $u+v = v+u$
- 2) For all u, v, w in V , $(u+v)+w = u+(v+w)$
- 3) There is an elt of V , called "0", so that for all v in V , $v+0 = v$.
- 4) For all v in V there exist w in V with
$$v+w = 0$$

5) For all v in V , $1v = v$. ②

6) For all a, b in \mathbb{R} and v in V , $(ab)v = a(bv)$

7) For all a in \mathbb{R} and u, v in V :

$$a(u+v) = au + av$$

8) For all a, b in \mathbb{R} and v in V , $(a+b)v = av + bv$

Example: \mathbb{R}^n with coordinate-wise addition and scalar mult.

[Check one rule, chosen by the class.]

Example:
$$\text{Mat}_{m \times n} = \left\{ \begin{array}{c} m \times n \text{ matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix} \text{ where } a_{ij} \text{ are in } \mathbb{R} \end{array} \right\}$$

where addition and scalar mult are again componentwise.

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 1 \end{pmatrix} + \underbrace{2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{pmatrix}}_{\begin{pmatrix} 2 & 2 & 4 \\ 0 & -2 & 6 \end{pmatrix}} = \begin{pmatrix} 3 & 2 & 7 \\ 0 & 3 & 7 \end{pmatrix}$$

Example: $\mathcal{F} = \{ \text{Continuous fns from } [-1, 1] \text{ to } \mathbb{R} \}$ (3)

$f+g$ is the fn where $(f+g)(x) = f(x) + g(x)$.

af is the fn where $(af)(x) = af(x)$

[Some aspects of vectors in 2 and 3d are not part of this definition (no dot product, for ex), however, many familiar properties do follow from these rules. For example,

Question: Is $0 \cdot v = 0$?

\uparrow in \mathbb{R} \uparrow in V .

Thm: If u, v, w are in a vector space V and $u+w = v+w$, then $u=v$.

Proof: By (4), there is a z in V with $w+z = 0$. So

$$u = u + 0 = u + (w+z) = (u+w) + z$$

(3)

$$= (v+w) + z = v + (w+z) = v + 0 = v$$

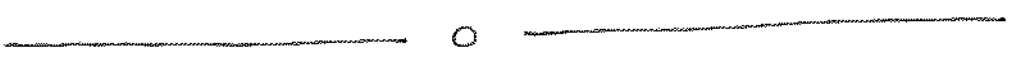
\uparrow
Hypothesis

Thm: If v is in a vector space V ,
then $0v = 0$ in V .

Proof: We have

$$\begin{aligned}
0v + 0v &= (0+0)v = 0v \stackrel{(3)}{=} 0v + 0 \\
&\stackrel{(1)}{=} 0 + 0v.
\end{aligned}$$

By the previous theorem, this gives
 $0v = 0$.



Related facts (see text and HW)

- a) The 0 vector is unique.
- b) The vector w with $v+w=0$ is unique;
we'll call it " $-v$ ". Note

$-v = (-1)v$ as

$$v + (-1)v \stackrel{(5)}{=} 1v + (-1)v \stackrel{(8)}{=} (1-1)v = 0 \cdot v = 0.$$

By above.
↓

Sometimes, will allow scalars other than \mathbb{R} , ⑤
most commonly the complex numbers $\mathbb{C} = \{a+bi\}$
where a, b are in \mathbb{R} and $i^2 = -1$.

Ex: $V = \mathbb{C}^2 = \{(z_1, z_2) \text{ where } z_i \text{ in } \mathbb{C}\}$

$$(2+i, 3) + \underbrace{(1+i)(1-i, 3i)}_{(2, 3i-3)} = (4+i, 3i)$$

[Useful for math math and physical applications.]

More generally, can define a vector space over any field \mathbb{F} , which is a set with operations $(+, \times, -, \div)$ satisfying a bunch of axioms.

Ex: Field of two elts $\{0, 1\}$ where

$0+0 = 0$
$0+1 = 1$
$1+0 = 1$
$1+1 = 0$

and

$0 \times 0 = 0$
$0 \times 1 = 0$
$1 \times 0 = 0$
$1 \times 1 = 1$

[Here, $-$ is the same as $+$ and \div is the same as \times
Finite fields are important in cryptography and coding theory, and are featured in Math 417.]

For the first part of the course we will always use \mathbb{R} for the scalars, but [FIS] uses the language of fields. ⑥

See Appendix C of [FIS] for more on fields.