Lecture 21: Definition of the determinant

Last time: $\det : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

Facts: ① $\det(A) \neq 0 \iff A$ is invertible

② $\det(AB) = \det(A)\det(B)$

③ $L_A : \mathbb{R}^n \to \mathbb{R}^n$ changes volumes by a factor of $|\det(A)|$.

④ Except when $n=1$, det is not linear.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have $\det(A) = ad - bc$.

For an $n\times n$ matrix A, let $\tilde{A}_{ij}$ denote the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad \tilde{A}_{11} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ $\quad \tilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\tilde{A}_{33} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Def: For $A \in M_{1\times 1}(\mathbb{R})$, set $\det(A) = A_{11}$.

For $A \in M_{n\times n}(\mathbb{R})$ with $n > 1$, inductively set

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

Note: Matches old def for $n=2$ as if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

we have

$$\det(A) = A_{11} \underbrace{\det(\tilde{A}_{11})}_{A_{22}} - A_{12} \underbrace{\det(\tilde{A}_{12})}_{A_{21}}$$

Ex:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \overset{A_{11}}{1} \cdot \det\overbrace{\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}}^{\tilde{A}_{11}} - \overset{A_{12}}{2} \cdot \det\overbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}^{\tilde{A}_{12}} + \overset{A_{13}}{3} \cdot \det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= 1 \cdot (-2) - 2 \cdot (-3) + 3 \cdot 1 = 7$$

Ex:

$$\det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix} = 4 \cdot \det(\tilde{A}_{11}) - 3 \cdot \det(\tilde{A}_{12}) + \cdots$$

$$= 4 \cdot 7 - \cdots$$

$$= 21.$$

[Will develop better methods to compute det later using our trusty row ops; using the def requires $n!$ multiplications...]

[To get to the properties mentioned at the beginning, must start small, those these are still useful...]

<u>Thm</u>: det is a linear function of the r^{th} row when all other rows are held fixed. In particular suppose $A, B, C \in M_{n\times n}(\mathbb{R})$ are the same except in row r where $a_r = b_r + k\, c_r$. (here a_r is the r^{th} row of A, etc.). Then

$$\det A = \det B + k \det C.$$

[Query: How does this not violate ④ above?]

<u>Proof</u>: First suppose $r = 1$. Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \left(B_{1j} + k C_{1j}\right) \det\left(\tilde{A}_{1j}\right)$$

also $= \tilde{B}_{1j}, \tilde{C}_{1j}$

$$= \sum_{j=1}^{n} (-1)^{j+1} B_{1j} \det\left(\tilde{B}_{1j}\right) + k \sum_{j=1}^{n} (-1)^{j+1} C_{1j} \det\left(\tilde{C}_{1j}\right)$$

$$= \det(B) + k \det(C).$$

In general, we induct on n.

<u>Base case</u>: $n = 1$ where det is actually linear.

<u>Inductive step</u>: Assume proven for $M_{n \times n}(\mathbb{R})$.

If $r = 1$ we are done by above, so assume $r > 1$. Then

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{j+1} \underbrace{A_{1j}}_{= B_{1j} = C_{1j}} \det\left(\tilde{A}_{1j}\right)$$

The same as $\tilde{B}_{ij}$ and $\tilde{C}_{ij}$ except in row $(r-1)$.

$$= \sum_{j=1}^{n+1} (-1)^{j+1} A_{1j} \left(\det\left(\tilde{B}_{1j}\right) + k \det\left(\tilde{C}_{1j}\right) \right)$$

$$= \det(B) + k \det(C).$$

$\blacksquare$

Thm: Suppose $A \in M_{n\times n}(\mathbb{R})$. For any r with $1 \le r \le n$ we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

Lemma: Suppose $B \in M_{n\times n}(\mathbb{R})$ where row r of B equal to e_j. Then $\det(B) = (-1)^{r+j} \det(\tilde{B}_{rj})$

Note: When $n=1$, we have $r=j=1$ and $B=(1)$. So for this to hold, let's define $\det(0\times 0 \text{ matrix}) = 1$.

Proof of theorem assuming the lemma:

Set $B_j \in M_{n\times n}(\mathbb{R})$ to be A with the r^{th} row replaced by e_j. Thus

$$(r^{th} \text{ row of } A) = \sum_{j=1}^{n} A_{rj} (r^{th} \text{ row of } B_j)$$

By the first theorem, we have

$$\det(A) = \sum_{j=1}^{n} A_{rj} \det(B_j)$$

$$= \sum_{j=1}^{n} A_{rj} (-1)^{r+j} \det\big(\underbrace{(\tilde{B_j})_{rj}}_{=\tilde{A}_{rj}}\big)$$

as required. $\blacksquare$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad r = 2$

$$\det(A) = -A_{12} \det(\tilde{A}_{12}) + A_{22} \det(\tilde{A}_{22}) - A_{23} \det(\tilde{A}_{23})$$

$$= (-1) \cdot \underbrace{\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}}_{\text{means "det of."}} + 0 \cdot \det(\tilde{A}_{22}) - 2 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (-1) \cdot (-1) - 2 \cdot (-3) = 7. \checkmark$$

<u>Next time:</u> Proof of the lemma.