Lecture 24: Determinant wrap-up.

Last lecture:

Thm: $A \in M_{n\times n}(\mathbb{R})$ is invertible iff it is the product of elementary matrices.

Thm: $A, B \in M_{n\times n}(\mathbb{R})$. Then $\det(AB) = \det(A)\det(B)$.

Thm: $A \in M_{n\times n}(\mathbb{R})$. Then $\det(A) \neq 0$ if and only if A is invertible.

Proof: If A is not invertible, saw last time that $\det(A) = 0$. So assume A is invertible. Then

$$\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1.$$

So $\det(A) \neq 0$. $\blacksquare$

Thm For $A \in M_{n\times n}(\mathbb{R})$, have $\det(A^t) = \det(A)$

Proof: By practice exam know

$$\operatorname{rank}(A) = \dim(\operatorname{RowSp}(A)) \underset{\substack{\uparrow \\ \text{Thm}}}{=} \dim(\overbrace{\operatorname{ColSp}(A)}^{=\operatorname{RowSp}(A^t)})$$

$$= \operatorname{rank}(A^t).$$

If A, A^t have rank $< n$, have $\det(A) = 0$ and $\det(A^t) = 0$ which match. If instead they are both invertible, by theorem A is the product of elementary matrices

$$A = E_1 E_2 \cdots E_k \cdots E_\ell$$

Now one can check that any elementary E_k satisfies $\det(E_k^t) = \det(E_k)$. Thus

$$\det(A^t) = \det(E_\ell^t E_{\ell-1}^t \cdots E_2^t E_1^t)$$
$$= \prod \det(E_k^t) = \prod \det(E_k) = \det(A)$$

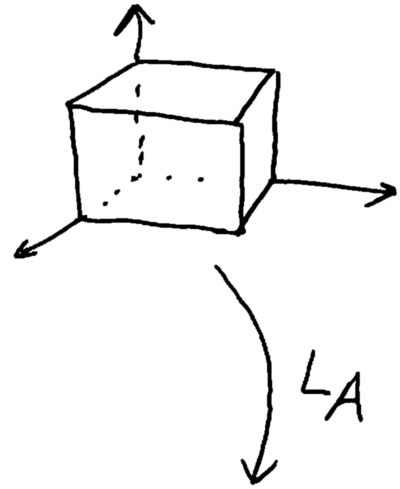
as required. ▨

[Two weeks ago, started with a list of 4 properties of det, the only one we're still missing is...]

<u>Thm</u>: Suppose $A \in M_{n\times n}(\mathbb{R})$. Then $\det(A)$ measures, with sign, how $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ changes volumes of objects. Specifically, if $S \subseteq \mathbb{R}^n$ is closed and bounded, then

$$\text{Volume}(L_A(S)) = |\det(A)| \text{ Volume}(S)$$

<u>Ex</u>: $S =$ unit (hyper) cube in $\mathbb{R}^n$ $= \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid 0 \le t_i \le 1\}$



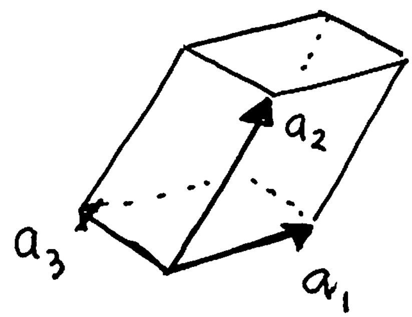
Then

$$L_A(S) = \left\{ \sum_{i=1}^{n} t_i a_i \;\middle|\; 0 \le t_i \le 1 \right\}$$

where a_i is the i^{th} column of A.

<u>Reason</u>: $(t_1, \dots, t_n) = \sum_{i=1}^{n} t_i e_i$

and $L_A(e_i) = i^{th}$ col of A.



By definition, something like ↗ is called an n-dimensional parallelopiped.

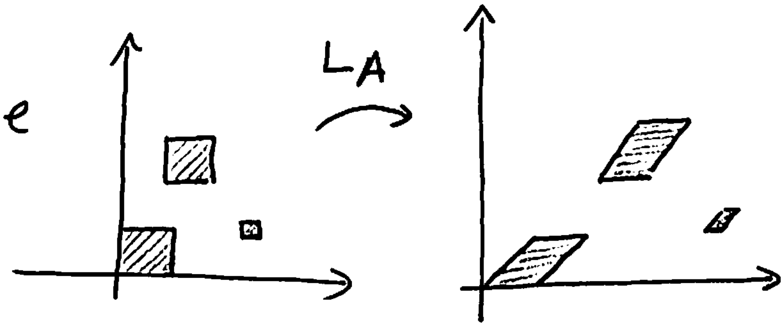
<u>Cor</u>: The volume of the parallelopiped determined by vectors $a_1, \dots, a_n$ in $\mathbb{R}^n$ is $\left| \det \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \right|$

$$= \left| \det \begin{pmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \end{pmatrix} \right|.$$

<u>Note</u>: For $n = 3$, this was the triple product $a_1 \cdot (a_2 \times a_3)$.

Sketch of ideas behind theorem.

- Any L_A distorts volume "uniformly."

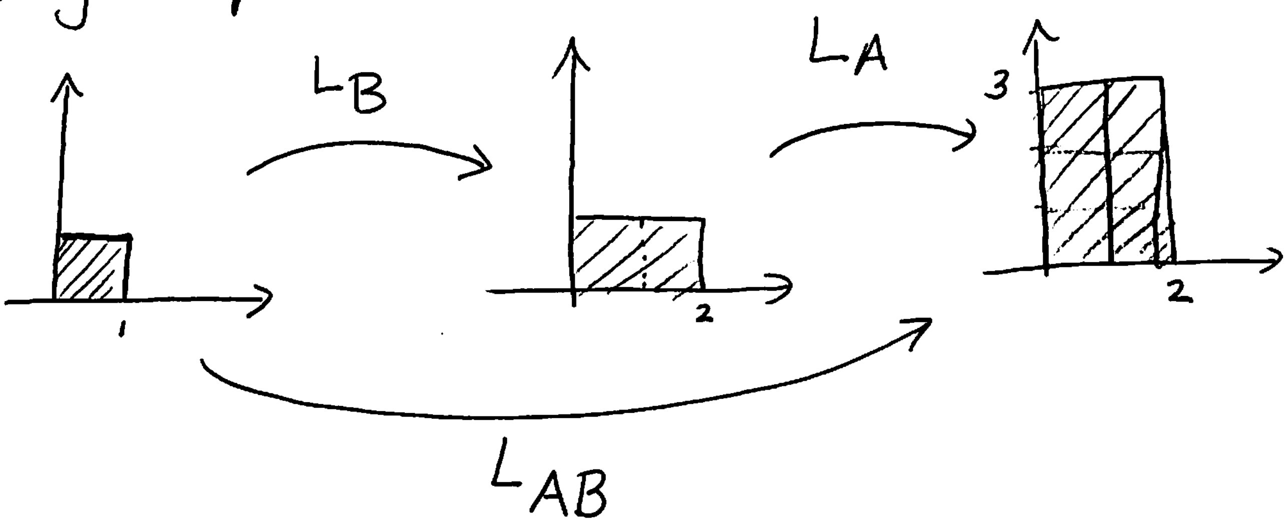


- Define $J(A)$ to be in $\mathbb{R}_{\geq 0}$ so that

$$\mathrm{Vol}(L_A(S)) = J(A) \cdot \mathrm{Vol}(S)$$

for all closed and bounded S.

- Study composition



So

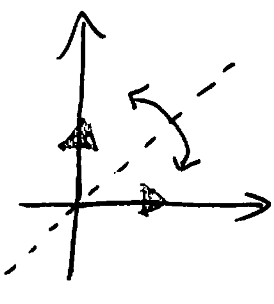
$$J(AB) = J(A)\,J(B)$$

- Show $J(E) = |\det(E)|$ for elementary matrices.

Combining with result that every invertible matrix is a product of elementary ones gives the theorem.

<u>What does L_E do?</u>

① $I_n \xrightarrow{R_r \leftrightarrow R_s} E$:   <u>Ex</u>: $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reflection in $y = x$



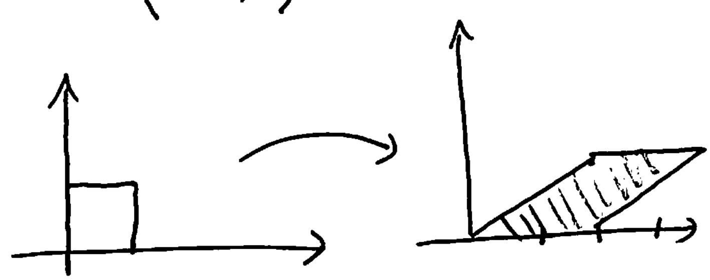
In general, E is reflection in the hyperplane $x_r = x_s$. ~~[illegible]~~ This preserves volumes, meshing with $\det(E) = -1$.

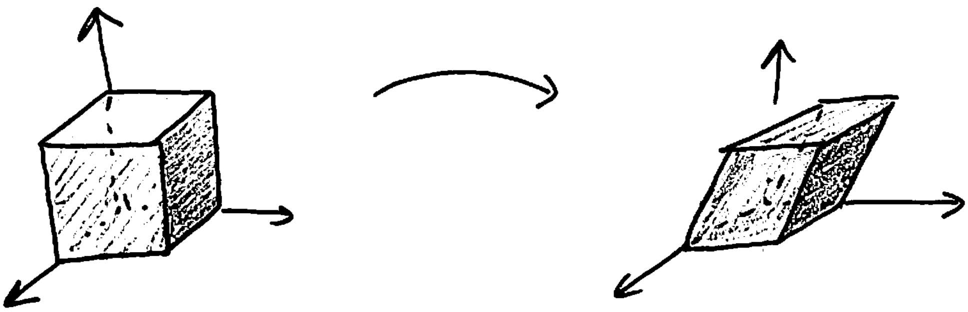
② $I_n \xrightarrow{cR_r} E$: just stretches by c in e_r direction,

$$\begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots \\ 0 & & & & & & 1 \end{pmatrix}$$

changing vol by $|c| = |\det(E)|$.

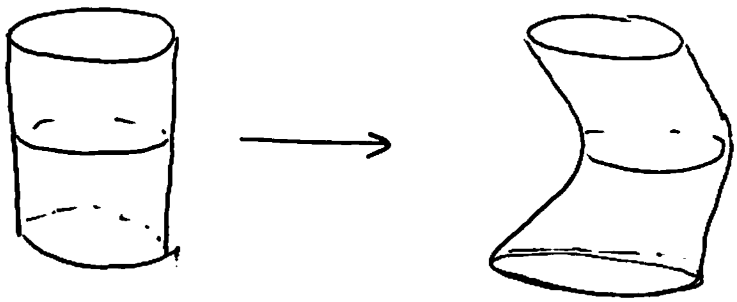
③ $I_n \xrightarrow{cR_s + R_r} E$:   <u>Ex</u>: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$





$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det = 1.$$

Doesn't change volume by Cavalieri's principle.



Cross-sectional area doesn't change

$\Rightarrow$ Volume doesn't change.

Basis for usual multivariable integration and works in all dimensions.

The other way to prove the theorem is change of coordinates...