Lecture 4: Linear combinations and systems of linear equations

[§1.4 of FIS] and [§SSLE of B]

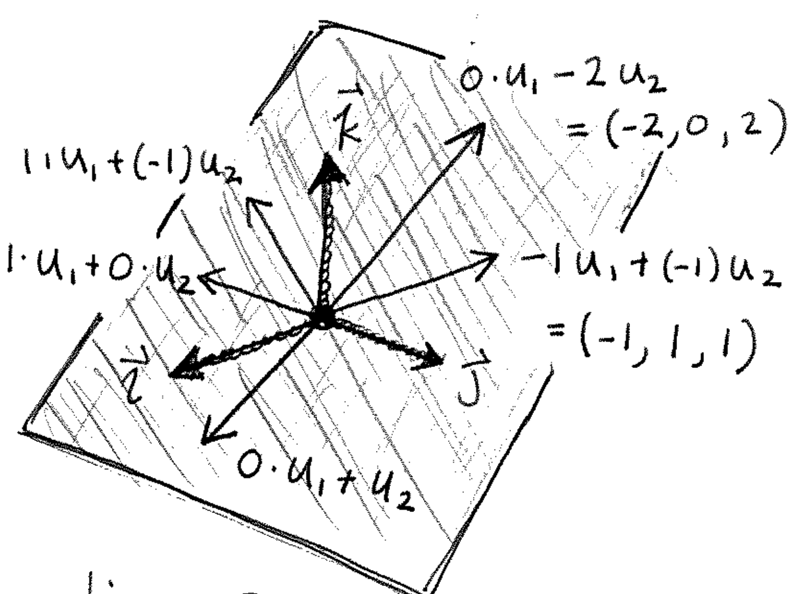
Last time: A subspace W of a vector space V is something where
(a) 0 is in W (b) W is closed under addition.
(c) W is closed under scalar mult.

A linear combination of vectors $u_1, u_2, \ldots, u_n$ in V is any vector of the form $a_1u_1 + a_2u_2 + \cdots + a_nu_n$ where the a_i are in $\mathbb{R}$.

Ex: $V = \mathbb{R}^3$

$u_1 = (0, -1, 0)$

$u_2 = (1, 0, -1)$



Some linear combinations.

The span of vectors $u_1, \dots, u_n$ in V is the set of all linear combinations of $u_1, \dots, u_n$.

Ex: $\text{span}(u_1, u_2) = \{ a_1 u_1 + a_2 u_2 \text{ for } a_1, a_2 \text{ in } \mathbb{R} \}$

$$= \{ (a_2 - a_1, -a_2) \}$$

$$= \text{plane } W \text{ in } \mathbb{R}^3 \text{ given by } \{ X + Z = 0 \}.$$

Thm: The span of any $u_1, \dots, u_n$ in V is a subspace of V.

Pf: (a) $0 = 0u_1 + \dots + 0u_n$

(b) $(a_1 u_1 + \dots + a_n u_n) + (b_1 u_1 + \dots + b_n u_n)$

$$= (a_1 + b_1) u_1 + \dots + (a_n + b_n) u_n$$

[Query: Why can I do this?]

(c) For c in $\mathbb{R}$ have

$$c \cdot (a_1 u_1 + \dots + a_n u_n) = (ca_1) u_1 + \dots + (ca_n) u_n .$$

$\blacksquare$

Ex: $V = \mathbb{R}^3$ $\quad u_1 = (1, 1, -1)$
$u_2 = (-1, 1, 2)$

$W = \text{span}(u_1, u_2)$

Q1: Find eqn for W of the form $c_1 x + c_2 y + c_3 z = 0$

Know $\quad c_1 \cdot 1 + c_2 \cdot 1 + c_3(-1) = 0$ $\quad$ since u_1, u_2
$c_1(-1) + c_2 \cdot 1 + c_3 \cdot (2) = 0$ $\quad$ are in W.

This gives the following system of eqns

$$\begin{aligned} c_1 + c_2 - c_3 &= 0 \\ -c_1 + c_2 + 2c_3 &= 0 \end{aligned} \xrightarrow[\text{to bottom}]{\text{Add top}} \begin{aligned} c_1 + c_2 - c_3 &= 0 \\ 2c_2 + c_3 &= 0 \end{aligned}$$

$$\xrightarrow[\text{to top}]{\text{Add bottom}} \begin{aligned} c_1 + 3c_2 \quad &= 0 \\ 2c_2 + c_3 &= 0 \end{aligned}$$

If we take $c_2 = -1$, get $c_1 = 3$, $c_3 = 2$.

So equation for plane is $3x - y + 2z = 0$.

[Double check!]

Q2: $v = (5, 1, -7)$ is in W as it satisfies the eqn. Thus it must be a linear combination of u_1 and u_2, i.e.

$$v = a_1 u_1 + a_2 u_2 = (a_1 - a_2, a_1 + a_2, -a_1 + 2a_2)$$

leading to a system of three equations:

$$\begin{aligned} a_1 - a_2 &= 5 \\ a_1 + a_2 &= 1 \\ -a_1 + 2a_2 &= -7 \end{aligned}$$

add $\rightsquigarrow$ $3a_2 = -6$

$\Rightarrow$ $a_2 = -2$

"implies"

and so $a_1 = 1 - a_2 = 3$. Check:

$$3 \cdot u_1 - 2u_2 = (3, 3, -3) + (2, -2, -4) = v \checkmark$$

Q3: $w = (6, 1, -7)$ is not in W. If we try to write w as a combination of u_1 and u_2, we are lead to

$$\begin{aligned} a_1 - a_2 &= 6 \\ a_1 + a_2 &= 1 \\ -a_1 + 2a_2 &= -7 \end{aligned}$$

which still leads to $a_2 = -2$ but $a_1 = 8$ by the 1^{st} eqn but $a_1 = 3$ by the second one. So there are no solutions to this system, which makes sense geometrically.

<u>System of Linear Equations:</u>

Variables: $x_1, x_2, \ldots, x_n$

Equations:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x &= b_m \end{aligned}$$

where a_{ij} and b_i are real numbers for $1 \le i \le m$, $1 \le j \le n$.

[So no x_i^2 much less a trig fn.]

<u>Basic tool to solve:</u>

Add two equations together, or add a multiple of one eqn to another:

<u>Ex</u>: For

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 3 \\ 3x_1 - 2x_2 + 6x_3 &= 1 \end{aligned} \quad ☆$$

might take $2(\text{Eqn } 1) + (\text{Eqn } 2) \rightsquigarrow 7x_1 = 7$

$$4x_1 + 2x_2 - 6x_3 = 6$$

Thus $x_1 = 1$.

<u>Goal</u>: Develop systematic method for solving such equations.

$$☆ \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 1 & -3 & 3 \\ 3 & -2 & 6 & 1 \end{array}\right)$$

This will be our basic computational tool for the first half of the symmester...

<u>Thm</u>: A linear system has either exactly one solution, no solutions, or infinitely many solutions.

Note that we saw this in the 3 examples today.