Math 418: HW 4 due Wednesday, February 23, 2022.

Webpage: http://dunfield.info/418
Office hours: Monday and Tuesday from 1:30–2:30pm; other times possible by appointment.

1. Let K/F be an algebraic extension. Suppose R is a *subring* contained in K which contains F. Prove that R is actually a *subfield* of K. Hint: First show that R is a vector space over F.

2. Prove that $\alpha = \cos(2\pi/5)$ is a constructable number. Use this to show that the regular 5-gon is constructable by straightedge and compass.

3. Find the splitting field K of $x^4 - 2$ over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$?

4. Find the splitting field K of $x^4 + x^2 + 1$ over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$?

5. Suppose K/F is the splitting field for a polynomial $f(x) \in F[x]$. Let $g(x) \in F[x]$ be irreducible. Show that if g has a root in K then it splits completely in $K[x]$.

   Hint: Consider the splitting field M/K of $g(x)$, where g is viewed as an element of $K[x]$. If $\alpha \in M$ is a root of g, first show that $K(\alpha)$ is the splitting field of $f(x)$ over $F(\alpha)$. Now try to use the uniqueness up to isomorphism parts of Theorems 8 and 27 in Chapter 13 of our textbook.