Math 418: HW 6 due Friday, March 25, 2022.

Webpage: http://dunfield.info/418

Office hours: Are special as this assignment is due on a Friday:

- Friday, March 11 at 11.
- Wednesday, March 23 at 1:30.
- Thursday, March 24 at 1:30.

The first five problems can be done using just what has been covered in class through Wednesday, March 9. Problem 6 requires the Fundamental Theorem of Galois Theory, which will stated at the end of lecture on March 21.

1. For $K = \mathbb{Q}(\sqrt[4]{2})$ and $F = \mathbb{Q}(\sqrt{2})$, explicitly determine all automorphisms of K/F.

2. Let k be a field, and $L = k(t)$ corresponding field of rational functions.

   (a) Prove that there is a unique $\sigma \in \mathrm{Aut}(L/k)$ with $\sigma(t) = t + 1$. Hint: Is $t + 1$ algebraic or transcendental over k?

   (b) Find the fixed field F of σ, i.e. compute $L_{\langle \sigma \rangle}$.

3. Determine the minimal polynomial over $\mathbb{Q}$ of $\gamma = 1 + \sqrt[3]{2} + \sqrt[3]{4}$.

4. Consider $K = \mathbb{Q}(\sqrt[8]{2}, i)$ from Example 3 of Section 14.2, and let $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$, and $F_3 = \mathbb{Q}(i\sqrt{2})$. Show that $\mathrm{Aut}(K/F_1) \cong Z_8$, $\mathrm{Aut}(K/F_2) \cong D_8$, and $\mathrm{Aut}(K/F_3) \cong Q_8$, where the groups are given in the notation of Section 5.3.

5. Suppose K is the splitting field over $\mathbb{Q}$ of a cubic polynomial $f(x) \in \mathbb{Q}[x]$. Show that if $\mathrm{Gal}(K/\mathbb{Q})$ is the cyclic group of order 3, then all the roots of f are real.

6. Let F be a field of characteristic $\neq 2$.

   (a) Suppose $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1, D_2, or $D_1 D_2$ is a square in F. Prove that K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group.

   (b) Conversely, suppose K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group. Prove that $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1, D_2, or $D_1 D_2$ is a square in F.