Math 418: HW 7 due Wednesday, April 6, 2022.

Webpage: http://dunfield.info/418

Office hours: Monday and Tuesday from 1:30–2:30pm; other times possible by appointment.

Note: The first three problems can be done using only what we've covered in class through Friday, March 25. The remaining three problems require the lecture of Monday, March 28 as well.

1. Find a primitive generator for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Be sure to justify your answer.

2. Let K/F be a Galois extension and $\alpha \in K$. Define the *norm* of α from K to F to be

$$N_{K/F}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$$

   (a) Prove that $N_{K/F}(\alpha) \in F$.

   (b) Prove that $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta)$. Thus, the norm gives a group homomorphism $K^\times \to F^\times$.

   (c) Prove that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ where $a \in F$ and $n = [K : F]$.

   (d) Let $K = F(\sqrt{D})$ be a quadratic extension of F. Show that $N_{K/F}(a + b\sqrt{D}) = a^2 - Db^2$, where $a, b \in F$.

   (e) Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Compute $N_{K/Q}$ for $\alpha = \sqrt[3]{2}$ and for $\zeta = \zeta_3$.

   Note: It is possible to define $N_{K/F}(\alpha)$ even when K/F is not Galois, see Problem #17 in Section 14.2 of our text.

3. As in the last problem, let K/F be a Galois extension with $n = [K : F]$ and $\alpha \in K$.

   (a) Let $f(x) = x^d + a_{d-1}x^{d-1} + \cdots a_1 x + a_0$ in $F[x]$ be the minimal polynomial of α over F. Prove that d divides n and that there are d distinct Galois conjugates of α in K, each of which is repeated n/d times in the formula for $N_{K/F}(\alpha)$. Use this to show $N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$.

   (b) For $\alpha \in K$, consider $T_\alpha \colon K \to K$ where $T_\alpha(\beta) = \alpha\beta$. As you know, this is an F-linear transformation; let A be the associated matrix with respect to some F-basis of K. If $K = F(\alpha)$, show that $\det(A) = N_{K/F}(\alpha)$. (In fact, this is true with no assumption on α.)

4. Consider $f(x) = (x^3 - 2)(x^3 - 3)$ in $\mathbb{Q}[x]$.

   (a) Determine the Galois group of $f(x)$ over $\mathbb{Q}$. That is, if K is the splitting field of f, compute $\mathrm{Gal}(K/\mathbb{Q})$. Note: You may assume that $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt[3]{3})$ are distinct subfields of K. (This can be verified by checking that $\sqrt[3]{3} - \sqrt[3]{2}$ is a root of $x^9 - 3x^6 + 165x^3 - 1$ and that the latter polynomial is irreducible.)

   (b) Find all subfields of K that contain $\mathbb{Q}(\zeta)$, where ζ is a primitive 3^{rd} root of unity.

5. Let θ be a root of $f(x) = x^3 - 3x + 1$. Show that $K = \mathbb{Q}(\theta)$ is a splitting field for f and that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. Express the other two roots of f explicitly in the form $a + b\theta + c\theta^2$ for $a, b, c \in \mathbb{Q}$.

6. Section 14.6 #19.