Math 418: Takehome 2 due Wednesday, April 20, 2022.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Cox, Little, and O'Shea, *Ideals, Varieties, and Algorithms*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in chapters 14-15 of Dummit and Foote, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it. While I believe all the questions are stated correctly, please contact me if you think something is fishy.

Office hours: While discussion of these specific problems will be limited to clarifying their statements, I will be happy to answer any broader questions about the course material durning my *special office hour*, Friday, April 15 at 4pm. There will be no office hours Monday, April 18 or Tuesday, April 19.

1. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$ with $n \geq 3$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$.

   (a) Let $\tau \in G$ be the restriction of complex conjugation. Find the element that τ corresponds to under the isomorphism $G \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$.

   (b) Let $K^+ = K \cap \mathbb{R}$. Prove that $K^+ = \mathbb{Q}(\alpha)$ where $\alpha = \zeta_n + \zeta_n^{-1}$.

2. Find the Galois group of $x^4 - 7$ over $\mathbb{Q}$ explicitly as a permutation group on the roots

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \left\{ \sqrt[4]{7}, -\sqrt[4]{7}, \sqrt[4]{7}i, -\sqrt[4]{7}i \right\}$$

   Clarification: Your answer should both give the isomorphism type of $\mathrm{Gal}(K/\mathbb{Q})$ and identify the explicit subgroup of $S_4 = \mathrm{SymmetricGroup}(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ which is its image under the map $\mathrm{Gal}(K/\mathbb{Q}) \to S_4$.

3. (a) For $I = (x^2 + 1) \subset \mathbb{R}[x]$, prove that I is maximal and that $\mathbf{V}(I) = \emptyset$.

   (b) Suppose that $I \subset \mathbb{R}[x]$ is maximal. Show that $\mathbf{V}(I)$ is either empty or a single point.

4. Let $R = \mathbb{C}[x_1, \ldots, x_n]$.

   (a) If $f \in R$ is an irreducible polynomial prove that the variety $\mathbf{V}(f) \subset \mathbb{C}^n$ is irreducible.

   (b) If $I \subset R$ is a proper ideal, prove that its radical is the intersection of all maximal ideals containing I.

5. Let V be an algebraic variety in k^n, and set $I = \mathbf{I}(V)$. Recall the coordinate ring of V is

$$\begin{aligned} k[V] &= \{f \colon V \to k \mid f \text{ is the restriction of a polynomial in } k[x_1, \ldots, x_n].\} \\ &= k[x_1, \ldots, x_n]/I \end{aligned}$$

In particular, two elements of $k[V]$ are the same if they agree at every point of V, even if nominally they come from different polynomials.

The ring $k[V]$ contains k as the subring of constant functions, coming from the polynomials with only a constant term. Thus it is a vector space over k. Prove that the following are equivalent.

(a) V is a finite set of points in k^n.

(b) $k[V]$ is finite-dimensional as a k-vector space.

Hints: For (a) ⇒ (b) consider $k[V]$ as a subspace of the vector space of *all* functions $f\colon V \to k$. For (b) ⇒ (a), for each coordinate, try to show that the set $\{a_i \mid (a_1, \ldots, a_n) \in V\}$ is finite.