Lecture 10: More on algebraic extensions

K/F field extension. An $\alpha \in K$ is algebraic over F if $\exists f(x) \in F[x]$ with $f(\alpha) = 0$. K/F is algebraic if every elt of K is algebraic over F.

Thm: $[K:F] < \infty \Rightarrow K/F$ is algebraic.

Thm: $\alpha, \beta \in K$ algebraic$/F$. Then $F(\alpha,\beta)/F$ is algebraic.

Cor: $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ alg over } \mathbb{Q}\}$ is a field.

Thm: $F \subseteq K \subseteq L$ fields. Then $[L:F] = [L:K][K:F]$

Finite Extension: K/F with $[K:F] < \infty$.

Cor: $F \subseteq K \subseteq L$. If L/K and K/F are both finite, so is L/F.

Cor: $F \subseteq K \subseteq L$. If L/K and K/F are both algebraic, so is L/F.

Pf: A given $\beta \in L$ is a root of some

$$p(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0 \in K[x].$$

Consider $F \subseteq F(\alpha_0) \subseteq F(\alpha_0, \alpha_1) \subseteq \cdots \subseteq F(\alpha_0, \ldots, \alpha_n) \subseteq$

$F(\alpha_0, \ldots, \alpha_n, \beta) = M$. Each of these extensions is simple and algebraic (as α_i are alg/F), and hence finite. By Cor, M/F is finite and hence algebraic. So β is alg over F. $\square$

[More consequences of this important thm.]

<u>Q</u>: Does one of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ contain the other? (Both $\subseteq \mathbb{R}$)

<u>A</u>: No. First $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3$. Now if $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[3]{2})$ would have $2 \cdot [\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}(\sqrt{2})] = 3$, which is silly.

<u>Def</u>: The <u>compositum</u> of fields $K_1, K_2 \subseteq L$ is the smallest subfield of L which contains both; it is denoted $K_1 K_2$.

<u>Ex</u>: $F(\alpha)F(\beta) = F(\alpha, \beta)$

<u>Ex</u>: $\mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[6]{2})$

Pf 1: $\mathbb{Q}(\sqrt[6]{2})$ contains $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ and $\sqrt{2}/\sqrt[3]{2} = 2^{1/2} \cdot 2^{-1/3} = 2^{1/6} = \sqrt[6]{2}$. ($\Rightarrow \mathbb{Q}(\sqrt[6]{2})$ is contained in the compositum.)

Pf 2: Any field containing $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ must have $[K:\mathbb{Q}]$ by 2 and 3 $\Rightarrow [K:\mathbb{Q}] \geq 6$. As $[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}] = 6$, it must be the compositum.

Thm: $F \subseteq K_1, K_2 \subseteq L$ with $[K_i:F] < \infty$. Then

$$[K_1K_2 : F] \leq [K_1:F][K_2:F]$$

Note: If $K_1 = F(\alpha)$ then this is easy as

$$[K_1K_2 : F] = \underbrace{[\overbrace{K_1K_2}^{K_1(\ )} : K_2]}_{= \deg m_{\alpha,K_2} \leq \deg m_{\alpha,F}}[K_2:F] \leq [K_1:F][K_2:F].$$

Pf: Let $\{\alpha_i\}$ be an F-basis for K_1.
Let $\{\beta_i\}$ be an F-basis for K_2.

Set $K = \{ \sum a_{ij} \alpha_i \beta_j \mid a_{ij} \in F \}$

Claim: $K_1K_2 = K$. This will suffice as $\dim_F K \leq \dim_F K_1 \cdot \dim_F K_2$.

Clearly $K_i \subseteq K \subseteq K_1 K_2$ so the issue is whether K is a subfield. Note K is closed under $+$ and also $\times$ since

$$(\alpha_i \beta_j)(\alpha_k \beta_\ell) = (\alpha_i \alpha_k)(\beta_j \beta_\ell) = \Big(\sum_m a_m \alpha_m\Big)\Big(\sum_n b_n \beta_n\Big)$$
$$= \sum_{m,n} \underbrace{a_m b_n}_{\in F} \alpha_m \beta_n$$

What about mult inverses?

Fix $\gamma \in K$. Consider $T : K \longrightarrow K$, $\delta \longmapsto \gamma\delta$, which is an F-linear transformation. As L is an int. domain,

finite-dim'l F-vector space.

$\ker T = \{0\} \Rightarrow T$ is onto as $\dim_F K < \infty$. In particular,

$\exists \delta \in K$ with $T(\delta) = 1$, i.e. $\delta\gamma = 1 \Rightarrow \gamma^{-1} = \delta \in K$.

So K is a subfield and hence $= K_1 K_2$ $\square$

The idea of field opperations as linear transformations will be very useful to us.

Ex: $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ $\qquad \gamma = 1+\sqrt{2}$

basis $\{1, \sqrt{2}\}$

$$T : \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2})$$
$$\delta \longmapsto \gamma \cdot \delta$$

$$1 \longmapsto \gamma = (1,1) \text{ in basis}$$
$$\sqrt{2} \longmapsto 2+\sqrt{2} = (2,1) \text{ in basis}$$

Matrix for T:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Matrix for $a+b\sqrt{2}$ with $a,b \in \mathbb{Q}$

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$$

This embeds $\mathbb{Q}(\sqrt{2})$ as a subring of $M_2(\mathbb{Q})$!